ACTA UNIVERSITATIS SZEGEDIENSIS

ACTA SCIENTIARUM MATHEMATICARUM

ADIUVANTIBUS

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REDIGIT

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TOMUS XXVII

FASC. 3—4

SZEGED, 1966

INSTITUTUM BOLYAIANUM UNIVERSITATIS SZEGEDIENSIS

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ACTA SCIENTIARUM MATHEMATICARUM

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27. KÖTET

3—4. FÜZET

SZEGED, 1966. DECEMBER

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On a process concerning inaccessible cardinals. II

By G. FODOR in Szeged

This paper is a continuation of reference I (see [1]), in which a process concerning inaccessible cardinals has been defined. In this paper we freely make use of the notations, definitions, and theorems of [1].

From now on, in the definition of the process, we start with strongly inaccessible initial numbers. This means that the values of the function $f_\eta(\alpha^{(0)}, \alpha^{(1)}, \dots, \alpha^{(\eta)})$ are strongly inaccessible numbers.

First we prove the following

Theorem 2. *If $\alpha = n_{\eta,\eta}(0)$ and $\eta < \alpha$ then the set of the ordinal numbers of the form $f_\eta(\alpha^{(0)}, \alpha^{(1)}, \dots, \alpha^{(\eta)}) < \alpha$ is non-stationary in α.*

Proof. We may assume by Theorem 1 of [1] that $\eta \geqq \omega$. Denote by $\gamma(\beta)$ the value $f_\eta(0, \dots, 0, \dots, \beta)$. As the first step we prove the following statement.

(j_1) *Suppose that $\beta \neq 0$. Then $\gamma(\beta)$ satisfies the equality*

$$\gamma(\beta) = f_\eta(0, \dots, 0, \dots, \gamma(\beta), \psi^{(\mu+1)}, \dots, \psi^{(\xi)}, \dots, \psi^{(\eta)})$$

for every $\mu < \eta$, provided that $\psi^{(\eta)} < \beta$ and $\psi^{(\xi)} < \gamma(\beta)$ for each ξ $(\mu + 1 \leqq \xi < \eta)$.

To prove this statement, we write η in the form $\eta = \omega\xi + n$, where $\xi \leqq \eta$ and $0 \leqq n < \omega$.

We distinguish the cases $n = 0$ and $n > 0$.

Case $n = 0$. We prove the following three statements, the third of which immediately implies (j_1):

(a) *If $\nu < \beta$ and $\tau < \eta$ then $\gamma(\beta)$ satisfies the equality*

$$\gamma(\beta) = f_\eta(0, \dots, 0, \dots, \overset{(\tau)}{\gamma(\beta)}, 0, \dots, 0, \dots, \nu).$$

(b) *If $\nu < \beta$, $\sigma < \xi$ and $0 < m < \omega$ then $\gamma(\beta)$ satisfies the equality*

$$\gamma(\beta) = f_\eta(0, \dots, 0, \dots, \gamma(\beta), \psi^{(\omega\sigma+1)}, \dots, \psi^{(\omega\sigma+l)}, \dots, \psi^{(\omega\sigma+m)}, 0, \dots, 0, \dots, \nu),$$

provided that $\psi^{(\omega\sigma+l)} < \gamma(\beta)$ for each l $(1 \leqq l \leqq m)$.

(c) *If $\nu < \beta$, $0 < \sigma < \xi$, $\varkappa < \omega\sigma$ and $0 \leqq m < \omega$ then $\gamma(\beta)$ satisfies the equality*

$$\gamma(\beta) = f_\eta(0, \dots, 0, \dots, \overset{(\varkappa)}{\gamma(\beta)}, 0, \dots, 0, \dots, \psi^{(\omega\sigma)}, \dots, \psi^{(\omega\sigma+l)}, \dots, \psi^{(\omega\sigma+m)}, 0, \dots, 0, \dots, \nu),$$

provided that $\psi^{(\omega\sigma+l)} < \gamma(\beta)$ for each l $(0 \leqq l \leqq m)$.

Ad (a): Since $\gamma(\beta)=f_\eta(0, \ldots, 0, \ldots, \beta)$, we have

(19) $$\gamma(\beta)\in Rf_\eta(\alpha^{(0)}, 0, \ldots, 0, \ldots, \beta).$$

It follows from the definition of $f_\eta(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\eta)})$ that

(20) $$Rf_\eta(\alpha^{(0)}, 0, \ldots, 0, \ldots, \varkappa) = \bigcap_{\nu<\varkappa} Rf_\eta(\alpha^{(0)}, 0, \ldots, 0, \ldots, \nu),$$

where $\varkappa$ is a limit number,

(21) $$Rf_\eta(\alpha^{(0)}, 0, \ldots, 0, \ldots, \varrho+1) = \bigcap_{\tau<\eta} Rf_\eta(0, \ldots, 0, \ldots, \alpha^{(\tau)}, 0, \ldots, 0, \ldots, \varrho),$$

and

(22) $$f_\eta(\alpha^{(0)}, 0, \ldots, 0, \ldots, 0, \overset{(\tau+1)}{1}, 0, \ldots, 0, \ldots, \nu) = \left(f_\eta(0, \ldots, 0, \ldots, \alpha^{(\tau)}, 0, \ldots, 0, \ldots, \nu)\right)'.$$

With the help of (19), (20) and (21) we obtain

(23) $$\gamma(\beta)\in Rf_\eta(\alpha^{(0)}, 0, \ldots, 0, \nu)$$

for every $\nu\leqq\beta$; moreover, (23) and (21) imply

(24) $$\gamma(\beta)\in Rf_\eta(0, \ldots, 0, \ldots, \alpha^{(\tau)}, 0, \ldots, 0, \ldots, \nu)$$

for every $\nu<\beta$ and for every $\tau<\eta$. From this we conclude that (a) is valid. For if not, then there are three ordinal numbers $\nu_0<\beta$, $\tau_0<\eta$ and $\varrho_0<\gamma(\beta)$ such that

$$\gamma(\beta) = f_\eta(0, \ldots, 0, \ldots, \overset{(\tau_0)}{\varrho_0}, 0, \ldots, 0, \ldots, \nu_0).$$

Hence, by (22), we have

$$\gamma(\beta)\notin Rf_\eta(\alpha^{(0)}, 0, \ldots, 0, \ldots, 0, \overset{(\tau_0+1)}{1}, 0, \ldots, 0, \ldots, \nu_0).$$

Thus, by the definition of $f_\eta(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\eta)})$, we obtain

$$\gamma(\beta)\notin Rf_\eta(0, \ldots, 0, \ldots, 0, \alpha^{(\tau_0+1)}, 0, \ldots, 0, \ldots, \nu_0),$$

which contradicts the fact that (24) is valid for every $\nu<\beta$ and $\tau<\eta$.

Ad (b): From (a) we get

$$\gamma(\beta) = f_\eta(0, \ldots, 0, \ldots, \overset{(\omega\sigma+m)}{\gamma(\beta)}, 0, \ldots, 0, \ldots, \nu)$$

for every $\nu<\beta$, $\sigma<\eta$ and for every m $(0<m<\omega)$. Hence

(25) $$\gamma(\beta)\in Rf_\eta(\alpha^{(0)}, 0, \ldots, 0, \ldots, \overset{(\omega\sigma+m)}{\gamma(\beta)}, 0, \ldots, 0, \ldots, \nu)$$

for every $\nu<\beta$, $\sigma<\eta$ and for every m $(0<m<\omega)$.

It follows from the definition of $f_\eta(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\eta)})$ that

(26) $$Rf_\eta(\alpha^{(0)}, 0, \ldots, 0, \ldots, \overset{(\omega\sigma+m)}{\gamma(\beta)}, 0, \ldots, 0, \ldots, \nu) = \\ = \bigcap_{\mu<\gamma(\beta)} Rf_\eta(0, \ldots, 0, \ldots, \alpha^{(\omega\sigma+m-1)}, \mu, 0, \ldots, 0, \ldots, \nu)$$

and

$$f_\eta(\alpha^{(0)}, 0, \ldots, 0, \ldots, \overset{(\omega\sigma+m)}{\mu+1}, 0, \ldots, 0, \ldots, \nu) = \tag{27}$$
$$=(f_\eta(0, \ldots, 0, \ldots, \alpha^{(\omega\sigma+m-1)}, \mu, 0, \ldots, 0, \ldots, \nu))'.$$

By (25) and (26) we have

$$\gamma(\beta)\in Rf_\eta(0, \ldots, 0, \ldots, \alpha^{(\omega\sigma+m-1)}, \mu, 0, \ldots, 0, \ldots, \nu) \tag{28}$$

for every $\mu<\gamma(\beta)$ and for every fixed $\nu<\beta$, $\sigma<\eta$ and m $(0<m<\omega)$. First we show that $\gamma(\beta)$ satisfies the equality

$$\gamma(\beta) = f_\eta(0, \ldots, 0, \ldots, \overset{(\omega\sigma+m-1)}{\gamma(\beta)}, \mu, 0, \ldots, 0, \ldots, \nu) \tag{29}$$

for every $\mu<\gamma(\beta)$ and for every fixed $\nu<\beta$, $\sigma<\eta$ and m $(0<m<\omega)$. If not, then there are two ordinal numbers $\mu_0<\gamma(\beta)$ and $\varrho_0<\gamma(\beta)$ such that

$$\gamma(\beta) = f_\eta(0, \ldots, 0, \ldots, \varrho_0, \overset{(\omega\sigma+m)}{\mu_0}, 0, \ldots, 0, \ldots, \nu).$$

Hence, by (27)

$$\gamma(\beta)\notin Rf_\eta(\alpha^{(0)}, 0, \ldots, 0, \ldots, \overset{(\omega\sigma+m)}{\mu_0+1}, 0, \ldots, 0, \ldots, \nu).$$

On the other hand it follows from this and the construction of $f_\eta(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\eta)})$ that

$$\gamma(\beta)\notin Rf_\eta(0, \ldots, 0, \ldots, \alpha^{(\omega\sigma+m-1)}, \mu_0+1, 0, \ldots, 0, \ldots, \nu),$$

which contradicts the fact that (28) holds for every $\mu<\gamma(\beta)$ and for every fixed $\nu<\beta$, $\sigma<\eta$ and m $(0<m<\omega)$. Thus we conclude that $\gamma(\beta)$ satisfies (29) for every $\nu<\beta$, $\sigma<\eta$ and for every m $(0<m<\omega)$.

Let now l be a natural number for which $0<l<m$. Assume that whenever $\nu<\beta$, $\sigma<\eta$, $0<m<\omega$ and $\psi^{(\omega\sigma+i)}<\gamma(\beta)$ $(i=l+1, \ldots, m)$ then

$$\gamma(\beta) = f_\eta(0, \ldots, 0, \ldots, \gamma(\beta), \psi^{(\omega\sigma+l+1)}, \ldots, \psi^{(\omega\sigma+i)}, \ldots, \psi^{(\omega\sigma+m)}, 0, \ldots, 0, \ldots, \nu).$$

Since $\gamma(\beta)=f_\eta(0, \ldots, 0, \ldots, \overset{(m\sigma+m-1)}{\gamma(\beta)}, \mu, 0, \ldots, 0, \ldots, \nu)$ for every $\nu<\beta$, $\sigma<\eta$, $0<m<\omega$ and for every $\mu<\gamma(\beta)$ it remains to prove that this assumption implies that whenever $\nu<\beta$ $\sigma<\eta$, $0<m<\omega$ and $\psi^{(\omega\sigma+i)}<\gamma(\beta)$ $(l\leqq i\leqq m)$ then

$$\gamma(\beta) = f_\eta(0, \ldots, 0, \ldots, \gamma(\beta), \psi^{(\omega\sigma+l)}, \ldots, \psi^{(\omega\sigma+i)}, \ldots, \psi^{(\omega\sigma+m)}, 0, \ldots, 0, \ldots, \nu).$$

It follows from the definition of $f_\eta(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\eta)})$ that, for given σ, ν, $0<m<\omega$, $\psi^{(\omega\sigma+1)}, \ldots, \psi^{(\omega\sigma+m)}$ the equalities

$$Rf_\eta(\alpha^{(0)}, 0, \ldots, 0, \ldots, \gamma(\beta), \psi^{(\omega\sigma+l+1)}, \ldots, \psi^{(\omega\sigma+m)}, 0, \ldots, 0, \ldots, \nu) = \tag{30}$$
$$= \bigcap_{\mu<\gamma(\beta)} Rf_\eta(0, \ldots, 0, \ldots, \alpha^{(\omega\sigma+l-1)}, \mu, \psi^{(\omega\sigma+l+1)}, \ldots, \psi^{(\omega\sigma+m)}, 0, \ldots, 0, \ldots, \nu)$$

and

$$f_\eta(\alpha^{(0)}, 0, \ldots, 0, \ldots, \mu+1, \psi^{(\omega\sigma+l+1)}, \ldots, \psi^{(\omega\sigma+m)}, 0, \ldots, 0, \ldots, \nu) =$$

(31)

$$= (f_\eta(0, \ldots, 0, \ldots, \alpha^{(\omega\sigma+l-1)}, \mu, \psi^{(\omega\sigma+l+1)}, \ldots, \psi^{(\omega\sigma+m)}, 0, \ldots, 0, \ldots, \nu)$$

hold.

By (30) and (31) we obtain for every $\mu<\gamma(\beta)$ and for any fixed $\nu<\beta$, $\sigma<\eta$, $0<m<\omega$ and $\psi^{(\omega\sigma+i)}<\gamma(\beta)$ $(l+1\leqq i\leqq m)$ that

(32)

$$\gamma(\beta)\in Rf_\eta(0, \ldots, 0, \ldots, \alpha^{(\omega\sigma+l-1)}, \mu, \psi^{(\omega\sigma+l+1)}, \ldots, \psi^{(\omega\sigma+i)}, \ldots, \psi^{(\omega\sigma+m)}, 0, \ldots, 0, \ldots, \nu).$$

Now we show for every $\mu<\gamma(\beta)$ and for any fixed $\nu<\beta$, $0<\eta$, $0<m<\omega$, $\psi^{(\omega\sigma+i)}<$ $<\gamma(\beta)$ $(l+1\leqq i\leqq m)$ that the ordinal number $\gamma(\beta)$ satisfies the equality

$$\gamma(\beta) = f_\eta(0, \ldots, 0, \ldots, \gamma(\beta), \mu, \psi^{(\omega\sigma+l+1)}, \ldots, \psi^{(\omega\sigma+i)}, \ldots, \psi^{(\omega\sigma+m)}, 0, \ldots, 0, \ldots, \nu).$$

In the contrary case there are two ordinal numbers $\mu_0<\gamma(\beta)$ and $\tau_0<\gamma(\beta)$ such that

$$\gamma(\beta) = f_\eta(0, \ldots, 0, \ldots, \tau_0, \mu_0, \psi^{(\omega\sigma+l+1)}, \ldots, \psi^{(\omega\sigma+m)}, 0, \ldots, 0, \ldots, \nu).$$

Hence, by (30), we have

$$\gamma(\beta)\notin Rf_\eta(\alpha^{(0)}, 0, \ldots, 0, \ldots, \mu_0+1, \psi^{(\omega\sigma+l+1)}, \ldots, \psi^{(\omega\sigma+m)}, 0, \ldots, 0, \ldots, \nu).$$

Consequently, by the definition of $f_\eta(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\eta)})$

$$\gamma(\beta)\notin Rf_\eta(0, \ldots, 0, \ldots, \alpha^{(\omega\sigma+l-1)}, \mu_0+1, \psi^{(\omega\sigma+l+1)}, \ldots, \psi^{(\omega\sigma+m)}, 0, \ldots, 0, \ldots, \nu).$$

Since $\gamma(\beta)$ is a limit number, we have $\mu_0+1<\gamma(\beta)$, which contradicts the fact that (32) holds for every $\mu<\gamma(\beta)$ and for any fixed $\nu<\beta$, $\sigma<\eta$, $0<m<\omega$ and $\psi^{(\omega\sigma+i)}<\gamma(\beta)$ $(l+1\leqq i\leqq m)$. Thus we may conclude that the statement (b) is true.

Ad (c): If $\nu<\beta$, $\sigma<\xi$ and $0<m<\omega$ then, by (b), $\gamma(\beta)$ satisfies the equality

$$\gamma(\beta) = f_\eta(0, \ldots, 0, \ldots, \gamma(\beta), \psi^{(\omega\sigma+1)}, \ldots, \psi^{(\omega\sigma+l)}, \ldots, \psi^{(\omega\sigma+m)}, 0, \ldots, 0, \ldots, \nu),$$

provided that $\psi^{(\omega\sigma+l)}<\gamma(\beta)$ for each l $(1\leqq l\leqq m)$. It follows from this, under the same conditions, that

$$\gamma(\beta)\in Rf_\eta(\alpha^{(0)}, 0, \ldots, 0, \ldots, \gamma(\beta), \psi^{(\omega\sigma+1)}, \ldots, \psi^{(\omega\sigma+l)}, \ldots, \psi^{(\omega\sigma+m)}, 0, \ldots, 0, \ldots, \nu).$$

Since, by the construction of $f_\eta(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\eta)})$

$$Rf_\eta(\alpha^{(0)}, 0, \ldots, 0, \ldots, \gamma(\beta), \psi^{(\omega\sigma+1)}, \ldots, \psi^{(\omega\sigma+m)}, 0, \ldots, 0, \ldots, \nu) =$$
$$= \bigcap_{\mu<\gamma(\beta)} Rf_\eta(\alpha^{(0)}, 0, \ldots, 0, \ldots, \mu, \psi^{(\omega\sigma+1)}, \ldots, \psi^{(\omega\sigma+m)}, 0, \ldots, 0, \ldots, \nu),$$

$$Rf_\eta(\alpha^{(0)}, 0, \ldots, 0, \ldots, \mu+1, \psi^{(\omega\sigma+1)}, \ldots, \psi^{(\omega\sigma+m)}, 0, \ldots, 0, \ldots, \nu) =$$
$$= \bigcap_{\tau<\omega\sigma} Rf_\eta(0, \ldots, 0, \ldots, \alpha^{(\tau)}, 0, \ldots, 0, \ldots, \mu, \psi^{(\omega\sigma+1)}, \ldots, \psi^{(\omega\sigma+m)}, 0, \ldots, 0, \ldots, \nu),$$

and

$$f_\eta(\alpha^{(0)}, 0, \ldots, 0, \ldots, 0, \overset{(\tau+1)}{1}, 0, \ldots, 0, \ldots, \nu) = (f_\eta(0, \ldots, 0, \ldots, \alpha^{(\tau)}, 0, \ldots, 0, \ldots, \nu))',$$

we can apply the method used in the proof of (a). Thus we obtain the proof of (c).

Case $n>0$. By the same argument as in the proof of (a) and (b) we obtain that $\gamma(\beta)$ satisfies the equality

$$\gamma(\beta)=f_\eta(0,\ldots,0,\ldots,\gamma(\beta),\psi^{(\omega\xi+1)},\ldots,\psi^{(\omega\xi+l)},\ldots,\psi^{(\omega\xi+n)})$$

for any $\psi^{(\omega\sigma+l)}<\gamma(\beta)$ $(1\leqq l\leqq n-1)$ and $\psi^{(\eta)}<\beta$.

Hence, by the argument used in the proof of (c), we obtain that $\gamma(\beta)$ satisfies the equality

$$\gamma(\beta)=f_\eta(0,\ldots,0,\ldots,\overset{(\varkappa)}{\gamma(\beta)},0,\ldots,0,\ldots,\psi^{(\omega\xi)},\ldots,\psi^{(\omega\xi+l)},\ldots,\psi^{(\omega\xi+n)})$$

for any $\varkappa<\omega\xi$, $\psi^{(\omega\xi+l)}<\gamma(\beta)$ $(0\leqq l\leqq n-1)$ and $\psi^{(\eta)}<\beta$.

From this, by the argument applied in the proof of (b), we conclude that $\gamma(\beta)$ satisfies the equality

$$\gamma(\beta)=f_\eta(0,\ldots,0,\ldots,\gamma(\beta),\psi^{(\omega\sigma+1)},\ldots,\psi^{(\omega\sigma+k)},\ldots,\psi^{(\omega\sigma+m)},0,\ldots$$
$$\ldots,0,\ldots,\psi^{(\omega\xi)},\ldots,\psi^{(\omega\xi+l)},\ldots,\psi^{(\eta)})$$

whenever $0<m<\omega$, $\sigma<\xi$, $\psi^{(\omega\sigma+k)}<\gamma(\beta)$ $(0<k\leqq m)$, $\psi^{(\omega\xi+l)}<\gamma(\beta)$ $(0\leqq l\leqq n-1)$, and $\psi^{(\eta)}<\beta$.

Finally, by the argument of the proof of (c), we obtain that $\gamma(\beta)$ satisfies the equality

$$\gamma(\beta)=f_\eta(0,\ldots,0,\ldots,\overset{(\varkappa)}{\gamma(\beta)},0,\ldots,0,\ldots,\psi^{(\omega\sigma)},\ldots,\psi^{(\omega\sigma+k)},\ldots$$
$$\ldots,\psi^{(\omega\sigma+m)},0,\ldots,0,\ldots,\psi^{(\omega\xi)},\ldots,\psi^{(\omega\sigma+l)},\ldots,\psi^{(\eta)})$$

whenever $m<\omega$, $\varkappa<\omega\sigma$, $\psi^{(\omega\sigma+k)}<\gamma(\beta)$ $(0\leqq k\leqq m)$, $\psi^{(\omega\xi+l)}<\gamma(\beta)$ $(0\leqq l\leqq n-1)$, and $\psi^{(\eta)}<\beta$. This immediately implies the statement (j_1) in the case $n>0$ too.

The same method can be used to prove the following statement:

(j_2) *Assume that* $\underline{\alpha}^{(\mu)},\ldots,\underline{\alpha}^{(\eta)}$ $(0<\mu\leqq\eta)$ *are given ordinal numbers and* $\underline{\alpha}^{(\eta)}\neq0$. *Then* $\gamma=f_\eta$ $(0,\ldots,0,\ldots,\underline{\alpha}^{(\mu)},\ldots,\underline{\alpha}^{(\eta)})$ *satisfies the equality*

$$\gamma=f_\eta(0,\ldots,0,\ldots,\gamma,\psi^{(\tau+1)},\ldots,\psi^{(\xi)},\ldots,\psi^{(\mu)},\underline{\alpha}^{(\mu+1)},\ldots,\underline{\alpha}^{(\eta)})$$

or every τ $(0\leqq\tau\leqq\mu)$ *provided that* $\psi^{(\mu)}<\underline{\alpha}^{(\mu)}$ *and* $\psi^{(\xi)}<\gamma$ *for each* ξ $(\tau+1\leqq\xi<\mu)$.

Now we proceed to prove the following statement:

(j_3) *Assume that* $\underline{\alpha}^{(0)},\ldots,\underline{\alpha}^{(\mu)},\ldots,\underline{\alpha}^{(\eta)}$ $(0\leqq\mu\leqq\eta)$ *are given ordinal numbers,* $\underline{\alpha}^{(0)}\neq0$ *and* $\underline{\alpha}^{(\mu)}\neq0$. *Then* $\gamma=f_\eta(\underline{\alpha}^{(0)},0,\ldots,0,\ldots,\underline{\alpha}^{(\mu)},\ldots,\underline{\alpha}^{(\eta)})$ *satisfies the equality*

$$\gamma=f_\eta(0,\ldots,0,\ldots,\gamma,\psi^{(\tau+1)},\ldots,\psi^{(\xi)},\ldots,\psi^{(\mu)},\underline{\alpha}^{(\mu+1)},\ldots,\underline{\alpha}^{(\eta)})$$

for every τ $(0\leqq\tau<\mu)$, *provided that* $\psi^{(\mu)}<\underline{\alpha}^{(\mu)}$ *and* $\psi^{(\xi)}<\gamma$ *for each* ξ $(\tau+1\leqq\xi<\mu)$.

Let us denote λ the ordinal number $\underline{\alpha}^{(\mu)}$. Consider first the case when μ is an ordinal number of the first kind. It follows from the definition of $f_\eta(\alpha^{(0)},\alpha^{(1)},\ldots,\alpha^{(\eta)})$ that

$$f(\alpha^{(0)},0,\ldots,0,\ldots,\varrho+1,\underline{\alpha}^{(\mu+1)}\ldots,\underline{\alpha}^{(\eta)})=(f_\eta(0,\ldots,0,\ldots,\alpha^{(\mu-1)},\varrho,\underline{\alpha}^{(\mu+1)},\ldots,\underline{\alpha}^{(\eta)})$$

for $\lambda=\varrho+1$ and

$$Rf_\eta(\alpha^{(0)},0,\ldots,0,\ldots,\lambda,\underline{\alpha}^{(\mu+1)},\ldots,\underline{\alpha}^{(\eta)})=\bigcap_{\nu<\lambda}Rf_\eta(0,\ldots,0,\ldots,\alpha^{(\mu-1)},\nu,\underline{\alpha}^{(\mu+1)},\ldots,\underline{\alpha}^{(\eta)})$$

for a limit number λ. These imply that for every $\nu<\lambda$

$$\gamma\in Rf_\eta(0,\ldots,0,\ldots,\alpha^{(\mu-1)},\nu,\underline{\alpha}^{(\mu+1)},\ldots,\underline{\alpha}^{(\eta)}).$$

Hence we easily conclude that

$$\gamma=f_\eta(0,\ldots,0,\ldots,\gamma,\nu,\underline{\alpha}^{(\mu+1)},\ldots,\underline{\alpha}^{(\eta)}).$$

Thus, by (j_2), we get (j_3) in the case where μ in an ordinal number of the first kind.

Suppose now that μ is a limit number. Then from the definition of $f_\eta(\alpha^{(0)},\alpha^{(1)},\ldots,\alpha^{(\eta)})$ we see that

$$Rf_\eta(\alpha^{(0)},0,\ldots,0,\ldots,\varrho+1,\underline{\alpha}^{(\mu+1)},\ldots,\underline{\alpha}^{(\eta)})=$$
$$=\bigcap_{\xi<\mu}Rf_\eta(0,\ldots,0,\ldots,\alpha^{(\xi)},0,\ldots,0,\ldots,\varrho,\underline{\alpha}^{(\mu+1)},\ldots,\underline{\alpha}^{(\eta)})$$

for $\lambda=\varrho+1$ and

$$Rf_\eta(\alpha^{(0)},0,\ldots,0,\ldots,\lambda,\underline{\alpha}^{(\mu+1)},\ldots,\underline{\alpha}^{(\eta)})=$$
$$=\bigcap_{\nu<\lambda}Rf_\eta(\alpha^{(0)},0,\ldots,0,\ldots,\nu,\underline{\alpha}^{(\mu+1)},\ldots,\underline{\alpha}^{(\eta)})$$

for a limit number λ. By a proof analogous to that of (b) and (c), we obtain (j_3) in the case where μ is a limit number.

Now we can prove the following statement:

(j_4) *Let $\{\varkappa_\zeta\}_{\zeta\leqq\sigma}$ $(\sigma\leqq\eta)$ be the strictly increasing sequence of the ordinal numbers $\varkappa\leqq\eta$ for which $\alpha^{(\varkappa)}\neq0$. Assume that $\varkappa_0=0$. Then $\gamma=f_\eta(\underline{\alpha}^{(0)},\underline{\alpha}^{(1)},\ldots,\underline{\alpha}^{(\eta)})$ satisfies the equality*

$$\gamma=f_\eta(0,\ldots,0,\ldots,\gamma,\psi^{(\tau+1)},\ldots,\psi^{(\xi)},\ldots,\psi^{(\varkappa_\zeta)},0,\ldots,0,\ldots,\underline{\alpha}^{(\varkappa_\zeta+1)},\ldots,\underline{\alpha}^{(\eta)})$$

for every ζ $(1\leqq\zeta\leqq\sigma)$ and for every τ $(0\leqq\tau\leqq\varkappa_\zeta)$, provided that $\psi^{(\varkappa_\zeta)}<\underline{\alpha}^{(\varkappa_\zeta)}$ and $\psi^{(\xi)}<\gamma$ for each ξ $(\tau+1\leqq\xi<\varkappa_\zeta)$.

Indeed, if (j_4) is true for a fixed ζ $(0<\zeta\leqq\sigma)$, then

$$\gamma=f_\eta(\gamma,0,\ldots,0,\ldots,\underline{\alpha}^{(\varkappa_\zeta)},0,\ldots,0,\ldots,\underline{\alpha}^{(\varkappa_\zeta+1)},\ldots,\underline{\alpha}^{(\eta)}).$$

If we apply (j_3) to $\underline{\alpha}^{(0)}=\gamma$, we obtain that

$$\gamma=f_\eta(0,\ldots,0,\ldots,\gamma,\psi^{(\tau+1)},\ldots,\psi^{(\xi)},\ldots,\psi^{(\varkappa_\zeta)},0,\ldots,0,\ldots,\underline{\alpha}^{(\varkappa_\zeta+1)},\ldots,\underline{\alpha}^{(\eta)})$$

for every τ $(0\leqq\tau\leqq\varkappa_\zeta)$, provided that $\psi^{(\varkappa_\zeta)}<\underline{\alpha}^{(\varkappa_\zeta)}$ and $\psi^{(\xi)}<\gamma$ for each ξ $(\tau+1\leqq\leqq\xi<\varkappa_\zeta)$. This proves the statement (j_4).

Now we proceed the proof of Theorem 2 by showing that the set

$$Rf_\eta(0,\ldots,0,\ldots,\beta)/\alpha \tag{3}$$

is non-stationary in α. We define a function g on $M=Rf_\eta(0,\ldots,0,\ldots,\beta)/\alpha$ by writing

$$g(f_\eta(0,\ldots,0,\ldots,\beta))=\beta.$$

Since $f_\eta(0, \ldots, 0, \ldots, \tau)$ is a strictly increasing function of the variable τ and for every $\beta < \alpha$ the inequality

$$\beta < f_\eta(0, \ldots, 0, \ldots, \beta)$$

holds, we obtain that the function g is strictly divergent and regressive on M. Therefore Theorem I (see [1]) implies that the set (33) is non-stationary in α.

Next we prove, by transfinite induction, the following statement.

(j_5) *For every* μ, $0 < \mu \leqq \eta$ *the set*

$$Rf_\eta(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\psi^{(\varrho)}, \ldots, \underline{\alpha}^{(\mu)}, \ldots, \underline{\alpha}^{(\eta)})/\alpha$$

is non-stationary in α, *where* $\underline{\alpha}^{(\mu)}, \ldots, \underline{\alpha}^{(\eta)}$ *are given ordinal numbers* $< \alpha$.

First we show that the set

$$N = Rf_\eta(\alpha_\xi^{(0)}, \underline{\alpha}^{(1)}, \ldots, \underline{\alpha}^{(\eta)})/\alpha$$

is non-stationary in α. We define a function g on N by writing

$$g\big(f_\eta(\alpha_\xi^{(0)}, \underline{\alpha}^{(1)}, \ldots, \underline{\alpha}^{(\eta)})\big) = \alpha_\xi^{(0)}.$$

From the definition of $\alpha_\xi^{(0)}(\underline{\alpha}^{(1)}, \ldots, \underline{\alpha}^{(\eta)})$ and $f_\eta(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\eta)})$, we obtain

$$\alpha_\xi^{(0)} < f_\eta(\alpha_\xi^{(0)}, \underline{\alpha}^{(1)}, \ldots, \underline{\alpha}^{(\eta)})$$

and

$$f_\eta(\alpha_\xi^{(0)}, \underline{\alpha}^{(1)}, \ldots, \underline{\alpha}^{(\eta)}) < f_\eta(\alpha_{\xi+1}^{(0)}, \underline{\alpha}^{(1)}, \ldots, \underline{\alpha}^{(\eta)}).$$

From these we infer that the function g is strictly divergent and regressive on N and, therefore, by Theorem I ([1]), we obtain that the set N is non-stationary in α.

Let ν be a given ordinal number and suppose that for every μ $(1 \leqq \mu < \nu)$ the set

$$Rf_\eta(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\psi^{(\varrho)}, \ldots, \underline{\alpha}^{(\mu)}, \ldots, \underline{\alpha}^{(\eta)})/\alpha$$

is non-stationary in α.

There are two cases:

a) ν is an ordinal number of the first kind, i.e. $\nu = \tau + 1$,

b) ν is an ordinal number of the second kind.

Case a): We show that the set

$$L = Rf_\eta(0, \ldots, 0, \ldots, \alpha_\varphi^{(\tau)}, \underline{\alpha}^{(\tau+1)}, \ldots, \underline{\alpha}^{(\eta)})/\alpha$$

is non-stationary in α. We define a function g on L by writing

$$g\big(f_\eta(0, \ldots, 0, \ldots, \alpha_\varphi^{(\tau)}, \underline{\alpha}^{(\tau+1)}, \ldots, \underline{\alpha}^{(\eta)})\big) = \alpha_\varphi^{(\tau)}.$$

From the definition of $\alpha^{(\tau)}(\underline{\alpha}^{(\tau+1)}, \ldots, \underline{\alpha}^{(\eta)})$ and $f_\eta(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\eta)})$ we obtain

$$\alpha_\varphi^{(\tau)} < f_\eta(0, \ldots, 0, \ldots, \alpha_\varphi^{(\tau)}, \underline{\alpha}^{(\tau+1)}, \ldots, \underline{\alpha}^{(\eta)})$$

and

$$f_\eta(0, \ldots, 0, \ldots, \alpha_\varphi^{(\tau)}, \underline{\alpha}^{(\tau+1)}, \ldots, \underline{\alpha}^{(\eta)}) < f_\eta(0, \ldots, 0, \ldots, \alpha_{\varphi+1}^{(\tau)}, \underline{\alpha}^{(\tau+1)}, \ldots, \underline{\alpha}^{(\eta)}).$$

From these we conclude that the function g is strictly divergent and regressive on

L, and, therefore, by Theorem I ([1]), we obtain that the set L is non-stationary in α. It follows from the construction of $f_\eta(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\eta)})$ that

$$f_\eta(0, \ldots, 0, \ldots, \alpha_\varphi^{(\tau)}, \underline{\alpha}^{(\tau+1)}, \ldots, \underline{\alpha}^{(\eta)}) \leqq f_\eta(\alpha_\xi^{(0)}, \ldots, \alpha_\psi^{(\varrho)}, \ldots, \alpha_\varphi^{(\tau)}, \underline{\alpha}^{(\tau+1)}, \ldots, \underline{\alpha}^{(\eta)}).$$

By our assumption, for given $\underline{\alpha}_\varphi^{(\tau)}$ the set

$$Rf_\eta(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \underline{\alpha}_\varphi^{(\tau)}, \underline{\alpha}^{(\tau+1)}, \ldots, \underline{\alpha}^{(\eta)})/\alpha$$

is non-stationary in α. On the other hand it is easy to verify that for any two different elements $\underline{\alpha}_\varphi^{(\tau)}$ and $\underline{\alpha}_\sigma^{(\tau)}$ the sets

$$Rf_\eta(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\psi^{(\varrho)}, \ldots, \underline{\alpha}_\varphi^{(\tau)}, \underline{\alpha}^{(\tau+1)}, \ldots, \underline{\alpha}^{(\eta)})/\alpha$$

and

$$Rf_\eta(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\psi^{(\varrho)}, \ldots, \underline{\alpha}_\sigma^{(\tau)}, \underline{\alpha}^{(\tau+1)}, \ldots, \underline{\alpha}^{(\eta)})/\alpha$$

have no common elements. Since the set of the first elements of the sets

$$Rf_\eta(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\psi^{(\varrho)}, \ldots, \alpha_\varphi^{(\tau)}, \underline{\alpha}^{(\tau+1)}, \ldots, \underline{\alpha}^{(\eta)})/\alpha$$

with $\alpha_\varphi^{(\tau)} \in A_{\tau,\eta}(\underline{\alpha}^{(\tau+1)}, \ldots, \underline{\alpha}^{(\eta)})$ is equal to L we obtain from Theorem II ([1]) that the union of these sets is non-stationary in α.

Case b): Put

$$Q_{\mu,\nu,\eta} = Rf_\eta(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \underline{\alpha}_\varkappa^{(\mu)}, \ldots, \underline{\alpha}_\varrho^{(\delta)}, \ldots, \underline{\alpha}^{(\nu)}, \ldots, \underline{\alpha}^{(\eta)})/\alpha,$$

where $\underline{\alpha}_\varrho^{(\delta)}$ is fixed for each δ $(\mu \leqq \delta < \nu)$. It is easy to see that

$$Q_{1,\nu,\eta} \subset Q_{2,\nu,\eta} \subset \ldots \subset Q_{\mu,\nu,\eta} \subset \ldots \; (\mu < \nu).$$

By the hypothesis the set $Q_{\mu,\nu,\eta}$ $(\mu < \nu)$ is non-stationary in α. Since $\mu < \nu \leqq \eta < \alpha$ by Theorem III ([1]), we obtain that the set

$$\bigcup_{\mu<\nu} Q_{\mu,\nu,\eta} = Rf_\eta(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\varphi^{(\mu)}, \ldots, \underline{\alpha}^{(\nu)}, \ldots, \underline{\alpha}^{(\eta)})/\alpha$$

is non-stationary in α. Thus the statement (j_5) is proved.

Since the set Rf_η $(0, \ldots, 0, \ldots, \beta)/\alpha$ is non-stationary in α, we obtain from (j_5) that the set

$$K = Rf_\eta(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\varrho^{(\eta)})/\alpha$$

is non-stationary in α.

Consider now an arbitrary element $\gamma = f_\eta(\underline{\alpha}^{(0)}, \underline{\alpha}^{(1)}, \ldots, \underline{\alpha}^{(\eta)})$ of K. Let $\{\varkappa_\zeta\}_{\zeta \leqq \sigma}$ $(\sigma \leqq \eta)$ be the strictly increasing sequence of the ordinal numbers $\varkappa$, $0 \leqq \varkappa \leqq \eta$, for which $\alpha^{(\varkappa)} \neq 0$. Let us denote by ζ_0 the smallest ordinal number $\zeta \leqq \sigma$ for which $\varkappa_\zeta \geqq 2$. Then the statements (j_1)—(j_5) imply that

$$\gamma = f_\eta(0, \ldots, 0, \ldots, \gamma, \psi^{(\tau+1)}, \ldots, \psi^{(\xi)}, \ldots, \psi^{(\varkappa_\zeta)}, \underline{\alpha}^{(\varkappa_\zeta+1)}, \ldots, \underline{\alpha}^{(\eta)}) \tag{34}$$

for every ζ $(\zeta_0 \leqq \zeta \leqq \sigma)$ and τ $(0 \leqq \tau \leqq \varkappa_\zeta)$, provided that $\psi^{(\varkappa_\zeta)} < \underline{\alpha}^{(\varkappa_\zeta)}$ and $\psi^{(\xi)} < \gamma$ for each ξ $(\tau + 1 \leqq \xi < \varkappa_\zeta)$.

Let us denote by $S_{\zeta,\tau}$, where $\zeta_0 \leqq \zeta \leqq \sigma$ and $0 \leqq \tau \leqq \varkappa_\zeta$, the set of the sequences

$$(\psi^{(\tau+1)}, \ldots, \psi^{(\xi)}, \ldots, \psi^{(\varkappa_\zeta)})$$

such that $\psi^{(\varkappa_\zeta)} < \alpha^{(\varkappa_\zeta)}$ and $\psi^{(\xi)} < \gamma$ for each ξ $(\tau + 1 \leqq \xi < \varkappa_\zeta)$. Since $\eta < \alpha$ and α is a strongly inaccessible initial number, the power of the set $S_{\zeta,\tau}$ is smaller than α.

It follows from the statement (j_5) that for any element $(\psi^{(\tau+1)}, \ldots, \psi^{(\xi)}, \ldots, \psi^{(\varkappa_\zeta)})$ of $S_{\zeta,\tau}$ the set

$$C(\psi^{(\tau+1)}, \ldots, \psi^{(\varkappa_\zeta)}) =$$
$$= Rf_\eta(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\varrho^{(\delta)}, \ldots, \gamma, \psi^{(\tau+1)}, \ldots, \psi^{(\varkappa_\zeta)}, \underline{\alpha}^{(\varkappa_\zeta+1)}, \ldots, \underline{\alpha}^{(\eta)})/\alpha$$

is non-stationary in α.

Since $\zeta \leqq \sigma \leqq \eta < \alpha$ and α is a strongly inaccessible initial number and hence the power of $S_{\zeta,\tau}$ is smaller than α, Theorem III ([1]) implies that the set

$$B(\gamma) = \bigcup_{\zeta \leqq \sigma} \bigcup_{\tau < \varkappa_\zeta} \bigcup_{\psi^{(\tau+1)} < \gamma} \cdots \bigcup_{\psi^{(\xi)} < \gamma} \cdots \bigcup_{\psi^{(\varkappa_\zeta)} < \underline{\alpha}^{(\varkappa_\zeta)}} C(\psi^{(\tau+1)}, \ldots, \psi^{(\xi)}, \ldots, \psi^{(\varkappa_\zeta)})$$

is non-stationary in α. On the other hand, by (34), the smallest element of the set $B(\gamma)$ is γ.

In this manner, with every element $\gamma = f_\eta(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\delta^{(\eta)})$ of K we have associated a non-stationary set $B(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\delta^{(\eta)})$ the smallest element of which is γ.

It only remains to prove that

$$\bigcup_{\gamma \in M} B(\gamma)$$

is non-stationary in α. Since K is non-stationary in α, the sets

$$B_0 = Rf_\eta(\alpha_\xi^{(0)}, \underline{\alpha}_\zeta^{(1)}, \ldots, \underline{\alpha}_\delta^{(\eta)})/\alpha,$$
$$B_1 = Rf_\eta(0, \alpha_\zeta^{(1)}, \underline{\alpha}_\psi^{(2)}, \ldots, \underline{\alpha}_\delta^{(\eta)})/\alpha,$$
$$\vdots$$
$$B_\mu = Rf_\eta(0, \ldots, 0, \ldots, \alpha_\varrho^{(\mu)}, \underline{\alpha}_\varphi^{(\mu+1)}, \ldots, \underline{\alpha}_\delta^{(\eta)})/\alpha,$$
$$\vdots$$

are non-stationary in α, where $\underline{\alpha}_\zeta^{(1)}, \ldots, \underline{\alpha}_\varrho^{(\mu)}, \ldots, \underline{\alpha}_\delta^{(\eta)}$ are fixed ordinal numbers $< \alpha$.

Let $\nu > 1$ be a given ordinal number, and suppose that for every μ $(1 \leqq \mu < \nu)$ the set

$$D(\underline{\alpha}_\varrho^{(\mu)}, \underline{\alpha}_\varphi^{(\mu+1)}, \ldots, \underline{\alpha}_\delta^{(\eta)}) = \tag{35}$$
$$= \bigcup_{\substack{\alpha_\xi^{(0)} \in A_{0,\eta} \\ \alpha_\xi^{(0)} < \alpha}} \cdots \bigcup_{\substack{\alpha_\psi^{(\vartheta)} \in A_{\vartheta,\eta} \\ \alpha_\psi^{(\vartheta)} < \alpha}} \cdots B(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\psi^{(\vartheta)}, \ldots, \underline{\alpha}_\varrho^{(\mu)}, \ldots, \underline{\alpha}_\delta^{(\eta)})$$

is non-stationary in α. We must prove that the set

$$D(\underline{\alpha}_\varphi^{(\nu)}, \ldots, \underline{\alpha}_\delta^{(\eta)}) = \tag{36}$$
$$= \bigcup_{\substack{\alpha_\xi^{(0)} \in A_{0,\eta} \\ \alpha_\xi^{(0)} < \alpha}} \cdots \bigcup_{\substack{\alpha_\psi^{(\vartheta)} \in A_{\vartheta,\eta} \\ \alpha_\psi^{(\vartheta)} < \alpha}} \cdots B(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\psi^{(\vartheta)}, \ldots, \underline{\alpha}_\varphi^{(\nu)}, \ldots, \underline{\alpha}_\delta^{(\eta)})$$

is non-stationary in α. It is easy to verify that the smallest element of (35) is $f_\eta(0, \ldots, 0, \ldots, \underline{\alpha}_\varrho^{(\mu)}, \ldots, \underline{\alpha}_\delta^{(\eta)})$. Thus the set of the first elements of the sets

$D(\alpha_\varrho^{(\mu)}, \underline{\alpha}_\varphi^{(\mu+1)}, \ldots, \underline{\alpha}_\delta^{(\eta)})$ with $\alpha_\varrho^{(\mu)} \in A_{\mu,\eta}(\underline{\alpha}_\varphi^{(\mu+1)}, \ldots, \underline{\alpha}_\delta^{(\eta)})$ is equal to B_μ. Suppose now that ν is a number of the first kind, i.e. $\nu = \vartheta + 1$. In this case Theorem IV ([1]) implies that the set

$$\bigcup_{\alpha_\sigma^{(\vartheta)} \in A_{\vartheta,\eta},\ \alpha_\sigma^{(\vartheta)} < \alpha} D(\alpha_\sigma^{(\vartheta)}, \underline{\alpha}_\varphi^{(\vartheta+1)}, \ldots, \underline{\alpha}_\delta^{(\eta)})$$

is non-stationary in α. Suppose now that ν is a limit number. For the proof of our statement it is sufficient to show that the set

$$\bigcup_{\mu<\nu} D(\underline{\alpha}_\varrho^{(\mu)}, \ldots, \underline{\alpha}_\delta^{(\eta)}) \qquad (\nu \leqq \eta < \alpha)$$

is non-stationary in α. But this follows from the hypothesis and from Theorem IV. Thus the proof of Theorem 2 is complete.

In an entirely analogous way it may be proved the following

Theorem 3. *If $\eta < \alpha$, $\mu < \eta$ and $\alpha = n_{\mu,\eta}(\underline{\alpha}^{(\mu)}, \ldots, \underline{\alpha}^{(\eta)})$ then the set of the ordinal numbers of the form $f_\eta(\alpha^{(0)}, \alpha^{(1)}, \ldots, \underline{\alpha}^{(\mu)}, \ldots \underline{\alpha}^{(\eta)}) < \alpha$ is non-stationary in α.*

We prove now the following

Theorem 4. *If $\alpha = f_\eta(\underline{\alpha}^{(0)}, \ldots, \underline{\alpha}^{(\xi)} \ldots, \underline{\alpha}^{(\eta)})$, $\eta < \alpha$, and $\underline{\alpha}^{(\xi)} < \alpha$ for each $\xi \leqq \eta$ then the set of the ordinal numbers of the form $f_\eta(\alpha^{(0)}\ \alpha^{(1)}, \ldots, \alpha^{(\eta)}) < \alpha$ is non-stationary in α.*

Proof. Let $\{\xi_\zeta\}_{\zeta \leqq \sigma}$ $(\sigma \leqq \eta)$ be the strictly increasing sequence of the ordinal numbers $\xi \leqq \eta$ for which $\underline{\alpha}^{(\xi)} \neq 0$.

Put

$$\gamma(\nu^{(\xi_\mu)}) = f_\eta(0, \ldots, 0, \ldots, \nu^{(\xi_\mu)}, \underline{\alpha}^{(\xi_\mu+1)}, \ldots, \underline{\alpha}^{(\eta)}),$$

where $\nu^{(\xi_\mu)} < \underline{\alpha}^{(\xi_\mu)}$ if $\mu = 0$ and $\nu^{(\xi_\mu)} \leqq \underline{\alpha}^{(\xi_\mu)}$ if $0 < \mu \leqq \sigma$.

First we show that the set

$$\{f_\eta(0, \ldots, 0, \ldots, \nu^{(\xi_0)}, \ldots, \underline{\alpha}^{(\xi_0+1)}, \ldots, \underline{\alpha}^{(\eta)}\}_{\nu(\xi_0) < \alpha(\xi_0)} \tag{37}$$

is non-stationary in α. Indeed, if $\underline{\alpha}^{(\xi_0)} = \underline{\nu}^{(\xi_0)} + 1$ then

$$f_\eta(0, \ldots, 0, \ldots, \underline{\nu}^{(\xi_0)}, \underline{\alpha}^{(\xi_0+1)}, \ldots, \underline{\alpha}^{(\eta)}) < f_\eta(0, \ldots, 0, \ldots, \underline{\alpha}^{(\xi_0)}, \ldots, \underline{\alpha}^{(\eta)});$$

moreover, if $\underline{\alpha}^{(\xi_0)}$ is a limit number, then

$$\lim_{\nu^{(\xi_0)} < \underline{\alpha}^{(\xi_0)}} f_\eta(0, \ldots, 0, \ldots, \nu^{(\xi_0)}, \underline{\alpha}^{(\xi_0+1)}, \ldots, \underline{\alpha}^{(\eta)}) < \alpha,$$

because $\underline{\alpha}^{(\xi_0)} < \alpha$ and α are regular. This implies that the set (37) is non-stationary in α.

Now we show that for every μ $(0 < \mu \leqq \sigma)$ the set

$$\{f_\eta(0, \ldots, 0, \ldots, \nu^{(\xi_\mu)}, \underline{\alpha}^{(\xi_\mu+1)}, \ldots, \underline{\alpha}^{(\eta)})\}_{\nu(\xi_\mu) \leqq \underline{\alpha}(\xi_\mu)} \tag{38}$$

is non-stationary in α. Indeed, if $\mu > 0$ then

$$f_\eta(0, \ldots, 0, \ldots, \underline{\alpha}^{(\xi_\mu)}, \ldots \underline{\alpha}^{(\eta)}) < f_\eta(0, \ldots, 0, \ldots, \underline{\alpha}^{(\xi_0)}, \ldots, \underline{\alpha}^{(\xi_\mu)}, \ldots, \underline{\alpha}^{(\eta)}),$$

on the other hand

$$f_\eta(0, \ldots, 0, \ldots, \underline{\alpha}^{(\xi_0)}, 0, \ldots, 0, \ldots, \underline{\alpha}^{(\xi_\mu)}, \ldots, \underline{\alpha}^{(\eta)}) \leqq f_\eta(\underline{\alpha}^{(0)}, \underline{\alpha}^{(1)}, \ldots, \underline{\alpha}^{(\eta)}) = \alpha.$$

Hence, for $\mu > 0$,

$$f_\eta(0, \ldots, 0, \ldots, \underline{\alpha}^{(\xi_\mu)}, \ldots, \underline{\alpha}^{(\eta)}) < \alpha.$$

Consequently, the set (38), where $0 < \mu \leqq \sigma$ is non-stationary in α.

We may suppose without loss of generality that $\xi_0 = 0$. [In virtue of (j_5) and the non-stationarity of the sets (38) with $0 < \mu \leqq \sigma$, the set

(39) $$\bigcup_{0<\mu\leqq\sigma} \; \bigcup_{\nu(\xi_\mu) \leqq \underline{\alpha}(\xi_\mu)} Rf_\eta(\alpha_\xi^{(0)}, \ldots, \alpha_\varphi^{(\delta)}, \ldots, \underline{\alpha}^{(\xi_\mu+1)}, \ldots, \underline{\alpha}^{(\eta)})/\alpha$$

is non-stationary in α. Applying to the set (39) the argument used for the set $Rf_\eta(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\varrho^{(\eta)})/\alpha$ after the proof of (j_5) in the proof of Theorem 2, we obtain Theorem 4.

Remark. If in the definition of the process we start with weakly inaccessible initial numbers then we can only prove Theorems 2 (see [1]), 3, and 4 for $\eta < \omega$.

We prove now the following

Theorem 5. *If α is the smallest ordinal number of η for which $\eta = f_\eta(0, \ldots, 0, \ldots, 1)$ then the set of the ordinal numbers of the form $f_\tau(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\tau)})$, where $\tau < \eta$, is non-stationary in α.*

Proof. First we show that the set $N = \{f_\varrho(0, \ldots, 0, \ldots, 1)\}_{\varrho<\alpha}$ is non-stationary in α.

Since α is the smallest ordinal number of η for which $\eta = f_\eta(0, \ldots, 0, \ldots, 1)$, the relation

(40) $$\varrho < f_\varrho(0, \ldots, 0, \ldots, 1)$$

holds for each $\varrho < \alpha$. By the definition of $f_\eta(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\eta)})$ we have

(41) $$f_\varrho(0, \ldots, 0, \ldots, 1) < f_{\varrho+1}(0, \ldots, 0, \ldots, 1).$$

Let us define the function g on the set N by writing

$$g(f_\varrho(0, \ldots, 0, \ldots, 1)) = \varrho.$$

It follows from (40) and (41) that the function g is strictly divergent and regressive on the set N. Therefore, by Theorem I ([1]), the set N is non-stationary in α.

Consider the set

(42) $$Rf_\varrho(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\delta^{(\varrho)})/\alpha \qquad (\varrho < \alpha).$$

Since, as by the definition of $f_\eta(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\eta)})$ the equalities

$$f_\alpha(\alpha^{(0)}, 0, \ldots, 0, \ldots, 0) = f_0(\alpha^{(0)}),$$
$$f_\alpha(\alpha^{(0)}, \alpha^{(1)}, 0, \ldots, 0, \ldots, 0) = f_1(\alpha^{(0)}, \alpha^{(1)}),$$
$$\vdots$$
$$f_\alpha(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\varrho)}, 0, \ldots, 0, \ldots, 0) = f_\varrho(\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(\varrho)}),$$
$$\vdots$$

hold, we may assume $\alpha_\delta^{(\varrho)} \geqq 1$ in (42).

With the help of (j_5) we get for given ϱ that the set

$$M_\varrho = Rf_\varrho(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\delta^{(\varrho)})/\alpha \qquad (\varrho < \alpha, \quad \alpha_\delta^{(\varrho)} \geqq 1)$$

is non-stationary in α. But the set $\{f_\varrho(0, \ldots, 0, \ldots, 1)\}_{\varrho<\alpha}$ of the first elements of the sets M_ϱ with $\varrho<\alpha$ is non-stationary in α. Therefore, making use of (j_5), the set

$$\bigcup_{\varrho<\alpha} Rf_\varrho(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\delta^{(\varrho)})/\alpha \qquad (\alpha_\delta^{(\varrho)} \geqq 1) \tag{43}$$

is non-stationary in α. Applying the same argument to the set (43) as in the proof of Theorem 2, after the proof of (j_5) for the set $Rf_\eta(\alpha_\xi^{(0)}, \alpha_\zeta^{(1)}, \ldots, \alpha_\varrho^{(\eta)})/\alpha$, Theorem 5 will be proved.

Reference

[1] G. Fodor, On a process concerning inaccessible cardinals. I, *Acta Sci. Math.*, **27** (1966), 111—124.

(Received February 1, 1965)

Über Halbgruppen, die ihre Ideale reproduzieren

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1. Einleitung

Bekanntlich nennt man eine Teilmenge I einer Halbgruppe H ein *Ideal* von H, falls $HI \subseteqq I$ und $IH \subseteqq I$ sind. Es ist nicht schwer solche Beispiele zu finden, wo das echte Enthaltensein $HI \subset I$ besteht. Man betrachte z.B. in der Halbgruppe

	a	b	c
a	b	b	a
b	b	b	b
c	b	b	c

das durch a erzeugte Hauptideal, welches wir (auch im folgenden) mit (a) bezeichnen: man kann leicht feststellen, daß dem Ideal (a) die Elemente a, b, aber dem Komplexenprodukt $H \cdot (a)$ nur b gehört.

Wir sagen, daß das Ideal I der Halbgruppe H *durch H von links (bzw. von rechts) reproduziert wird*, falls $HI = I$ (bzw. $IH = I$) ist; wenn beide Gleichungen bestehen, so sagen wir einfach, daß *I durch H reproduziert wird.* In dieser Arbeit werden wir notwendige und hinreichende Bedingungen dafür angeben, daß jedes Ideal einer Halbgruppe von einer oder von beiden Seiten durch die Halbgruppe reproduziert wird.

2. Vorbemerkungen

Man sieht sofort, daß es genügt, das Problem nur für Hauptideale zu untersuchen. Sind nämlich jedes Hauptideal der Halbgruppe H von links durch H reproduziert, so ergibt sich

$$HI = H \bigcup_{a \in I} (a) = \bigcup_{a \in I} H \cdot (a) = \bigcup_{a \in I} (a) = I$$

für jedes Ideal I von H.

Es sei vorausgesetzt, daß die Halbgruppe H die genannte Eigenschaft besitzt. Dann gilt, insbesondere, $HH = H$, da H ein Ideal von sich selbst ist. Das bedeutet, daß *in einer solchen Halbgruppen jedes Element zerlegbar ist,* d.h. jedes $h \in H$ in der Form $h = xy$ $(x, y \in H)$ dargestellt werden kann. Die Zerlegbarkeit aller Elemente ist also eine triviale notwendige Bedingung, die aber im allgemeinen keine hinreichende ist (s. das oben gegebene Beispiel). Für gewisse Klassen der Halbgruppen kann dieselbe Bedingung auch hinreichend sein (Korollar zu Satz 1).

Andererseits ist offensichtlich *hinreichend, daß jedes Element ein linksseitiges relatives Einselement besitze.* Dabei nennt man e ein linksseitiges relatives Einselement von h, wenn $eh=h$ ist. Wir werden sehen, daß diese Bedingung im allgemeinen nicht, für gewisse wichtige Klassen der Halbgruppen aber auch notwendig ist (Beispiel in § 5, bzw. Satz 2, Korollar 2 zu Satz 3 und Korollar zu Satz 5).

3. Zwei Spezialfälle

Zuerst untersuchen wir das Problem bezüglich zweier speziellen Klassen der Halbgruppen.

Satz 1. *Ist in einer kommutativen Halbgruppe H jedes Element zerlegbar, dann wird jedes Primideal von H durch H reproduziert.*

Korollar. *Ist in einer kommutativen Halbgruppe jedes Element zerlegbar und jedes Ideal prim, dann wird jedes Ideal durch die Halbgruppe reproduziert.*

Vor dem Beweis bemerken wir, daß die Voraussetzungen des Satzes sicher (und die Konklusion trivialerweise) erfüllt sind, wenn H eine Gruppe ist.

Beweis. Man betrachte ein Primideal I von H und ein beliebiges Element i aus I. Ist jedes Element von H zerlegbar, so findet sich ein Elementepaar x, y in H, so daß $i=xy$ ist. Aus $xy\in I$ folgt aber, daß entweder x oder y dem Ideal I gehört; wegen der Kommutativität von H dürfen wir $y\in I$ annehmen. Dann gilt aber

$$i=xy\in HI,$$

woraus $I\subseteq HI$ folgt. Andererseits ist $I\supseteq HI$.

In [2] haben wir eine Halbgruppe H *idealgeordnet* genannt, falls aus $a\in(b)$ und $b\in(a)$ $(a, b\in H)$ immer $a=b$ folgt. Für solche Halbgruppen gilt

Satz 2. *In einer idealgeordneten Halbgruppe H werden alle Ideale von links (dann und) nur dann durch H reproduziert, wenn jedes Element von H ein linksseitiges relatives Einselement hat.*

Beweis. Nach den Vorbemerkungen genügt zu zeigen, daß ein Hauptideal (a) einer idealgeordneten Halbgruppe H von links nur dann durch H reproduziert wird, wenn es ein $q(\in H)$ mit $a=qa$ gibt.

Es sei H eine solche Halbgruppe und a ein beliebiges Element von H. Aus $(a)=H\cdot(a)$ folgt, daß sich das Element a als ein Produkt

$$a=qb \qquad (q\in H,\ b\in(a)) \tag{1}$$

darstellen läßt. Daraus ergibt sich

$$a\in(b). \tag{2}$$

Da H idealgeordnet ist, erhält man nach (1) und (2), daß $a=b$ ist. Das bedeutet, wieder nach (1), daß $a=qa$ ist, was zu beweisen war.

4. Notwendige und hinreichende Bedingung

Es sei H eine beliebige Halbgruppe und 1 ein Symbol, das kein Element von H bedeutet. Die Vereinigungsmenge $H \cup 1$ bildet, bezüglich der Verknüpfungsdefinition (s. [1], Seite 4)

$$1 \cdot 1 = 1, \quad h \cdot 1 = 1 \cdot h = h \qquad (h \in H)$$

eine Erweiterungshalbgruppe von H. Auf Grund dieser Konstruktion definiert man H^1 wie folgt:

$$H^1 = \begin{cases} H, & \text{falls } H \text{ ein Einselement hat;} \\ H \cup 1 & \text{sonst.} \end{cases}$$

Mit Hilfe dieses Begriffes können wir den Hauptresultat der vorliegender Arbeit folgendermaßen formulieren:

Satz 3. *Alle Ideale einer Halbgruppe H werden durch H dann und nur dann von links reproduziert, wenn jedes Element a von H im Komplexenprodukt HaH^1 enthalten ist.*

Korollar 1. *Alle Ideale einer Halbgruppe H werden durch H dann und nur dann reproduziert, wenn jedes Element a von H in HaH enthalten ist.*

Korollar 2. *Alle Ideale einer kommutativen Halbgruppe H werden durch H dann und nur dann reproduziert, wenn jedes Element von H ein relatives Einselement hat.*

Beweis. Aus den trivialen Gleichungen $HH^1 = H^1H = H$ folgt, daß jeder der Komplexe HaH^1, H^1aH, HaH ein Ideal von H bildet. Eben darum stimmt HaH^1, H^1aH, bzw. HaH mit (a) überein, falls das Element a im entsprechenden Komplex enthalten ist. Daraus ergibt sich, daß das Enthaltensein $a \in HaH^1$ $(a \in H)$ mit der Gleichung $(a) = HaH^1$ gleichbedeutend ist, und dasselbe auch für die Komplexe H^1aH und HaH gilt. Auf Grund dieser Bemerkung kann man den Satz folgendermaßen beweisen:

Bekanntlich ist $(a) = H^1aH^1$, woraus sich

$$H(a) = H(H^1aH^1) = (HH^1)aH^1 = HaH^1$$

ergibt. Deshalb gilt $(a) = H(a)$ genau dann, wenn $(a) = HaH^1$, d.h. $a \in HaH^1$ ist. Damit ist der Beweis des Satzes erbracht.

Schreibt man im Satz 3 statt „links" das Wort „rechts", so soll er „HaH^1" durch den Ausdruck „H^1aH" ersetzen. Um Korollar 1 zu beweisen, haben wir deshalb zu zeigen, daß die beiden Relationen $a \in HaH^1$ und $a \in H^1aH$ mit der einzigen $a \in HaH$ gleichbedeutend sind.

Aus $a \in HaH$ folgen $a \in HaH^1$ und $a \in H^1aH$, denn HaH^1, $H^1aH \supset HaH$.

Andererseits setze man $a \in HaH^1$ und $a \in H^1aH$ voraus. Dann ist $(a) = HaH^1 = H^1aH$, woraus sich

$$(a) = HaH^1 = HH^1aH^1 = H(H^1aH) = HaH,$$

d.h. $a \in HaH$ ergibt.

Nunmehr betrachten wir eine kommutative Halbgruppe H. Nach der letzten Vorbemerkung in § 2 haben wir nur die Notwendigkeit der Bedingung im Korollar 2 zu beweisen. Dementsprechend setzen wir voraus, daß H alle ihre Ideale reproduziert. Nach Korollar 1 ist dann jedes $a \in H$ in HaH enthalten, woraus

$$a = xay = (xy)a \qquad (x, y \in H)$$

folgt; d. h. ist xy ein relatives Einselement von a. Damit ist auch Korollar 2 bewiesen.

Wir wollen den Satz für einige speziellen Klassen der Halbgruppen anwenden. Man nennt eine Halbgruppe H

regulär,	wenn	$a = axa$
linksregulär,	wenn	$a = xa^2$
rechtsregulär,	wenn	$a = a^2x$
intraregulär,	wenn	$a = xa^2y$

für jedes $a \in H$ mit geeigneten $x, y \in H$ gilt ([1], Seite 121). Aus diesen Definitionen folgt, daß in einer regulären Halbgruppe H ebenso $a \in Ha$, wie $a \in aH$, in einer links- oder rechtsregulären, bzw. intraregulären Halbgruppe H aber $a \in HaH$ für jedes $a \in H$ gilt. Für reguläre Halbgruppen ergibt sich also unmittelbar aus der Definition, für die übrigen drei Halbgruppenklassen aber nach Satz 3 die Behauptung des folgenden Satzes:

Satz 4. *Jedes Ideal einer regulären (oder linksregulären, rechtsregulären, bzw. intraregulären) Halbgruppe wird durch die Halbgruppe reproduziert.*

5. Der Fall endlicher Halbgruppen

Für endliche Halbgruppen erhalten wir dieselbe notwendige und hinreichende Bedingung wie für die kommutativen. Es gilt nämlich

Satz 5. *Ist jedes Element h einer endlichen Halbgruppe H in HhH^1 enthalten, so hat jedes Element ein linksseitiges relatives Einselement.*

Korollar. *Alle Ideale einer endlichen Halbgruppe H werden durch H (dann und) nur dann von links reproduziert, wenn jedes Element von H ein linksseitiges relatives Einselement hat.*

Beweis. Man betrachte eine Halbgruppe H, in der $h \in HhH^1$ für jedes $h \in H$ gilt, und setze voraus, daß es ein $a \in H$ gibt, welches kein linksseitiges relatives Einselement hat. Wegen $a \in HaH^1$ läßt sich dann a in der Form

$$a = c_1 a d_1 \qquad (c_1, d_1 \in H) \tag{7}$$

darstellen, wobei $c_1 a \neq a$ ist. Auch das Element $c_1 a$ hat kein linksseitiges relatives Einselement: wäre nämlich e ein solches, so ergäbe sich nach (7)

$$ea = \big(e(c_1 a)\big)d_1 = (c_1 a)d_1 = a,$$

d.h. wäre e auch bezüglich a ein linksseitiges relatives Einselement. Jedoch ist, nach den Voraussetzungen, c_1a in $H(c_1a)H^1$ enthalten, so daß Elemente c_2, d_2 in H existieren sollen, mit denen c_1a in der Form

$$c_1a = c_2(c_1a)d_2 \qquad (c_2, d_2 \in H)$$

entsteht; gleichzeitig sind c_2c_1a, c_1a und a paarweise verschieden.

Durch die gleiche Überlegung kann man eine unendliche Folge

$$a, c_1a, c_2c_1a, \ldots, c_nc_{n-1}\ldots c_2c_1a, \ldots$$

paarweise verschiedener Elemente von H bekommen. Damit haben wir erhalten, daß eine solche Halbgruppe unendlich sein soll; folglich ist die Behauptung des Satzes richtig. Das Korollar läßt sich daraus nach Satz 3 unmittelbar schließen.

Wir haben noch zu untersuchen, ob es Halbgruppen gibt, die ihre Ideale von links reproduzieren, aber Elemente ohne linksseitige relative Einselemente besitzen. Wir zeigen, daß jede rechtsreguläre, aber nicht intrareguläre Halbgruppe eine solche ist. Es gilt nämlich der

Satz 6. *Hat jedes Element einer rechtsregulären Halbgruppe ein linksseitiges relatives Einselement, so ist die Halbgruppe auch intraregulär.*

Korollar. *Jede endlich rechts- oder linksreguläre Halbruppe ist auch intraregulär.*

Beweis. Sind die Voraussetzungen des Satzes erfüllt, so gibt es zu jedem Element a der Halbgruppe Elemente x und y, so daß

$$xa = a \quad \text{und} \quad a = a^2y$$

ist. Daraus ergibt sich aber

$$a = a^2y = xa^2y,$$

womit der Satz schon bewiesen ist.

Schließlich folgt das Korollar aus den Sätzen 4, 3, 5 und 6; übrigens kann man dieses Korollar durch den Gedankengang des Beweises zu Satz 5 auch unmittelbar gewinnen.

Literatur

[1] A. H. Clifford—G. B. Preston, *The algebraic theory of semigroups*, Bd. I (Providence, 1961).
[2] G. Szász, Über eine Quasiordnung in Halbgruppen, *Publicationes Mathematicae, Debrecen* (zu erscheinen).

(Eingegangen am 22. November 1965)

Об операторах с ядерными мнимыми компонентами

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Пусть A — вполне непрерывный линейный оператор, действующий из гильбертова пространства $\mathfrak{H}$ в гильбертово пространство $\mathfrak{G}$. Обозначим через $\omega_1, \omega_2, \ldots$ отличные от нуля собственные числа оператора $(A^*A)^{\frac{1}{2}}$, занумерованные в порядке убывания, и через n_j—кратность числа ω_j. Члены последовательности

$$s_j = s_j(A) \qquad (j = 1, 2, \ldots, r_A; \quad r_A \leqq \infty)$$

где

$$s_1 = s_2 = \ldots = s_{n_1} = \omega_1, \quad s_{n_1+1} = s_{n_1+2} = \ldots = s_{n_1+n_2} = \omega_2, \ldots$$

называются *сингулярными* числами оператора A. Если $\mathfrak{H} = \mathfrak{G}$ и $\sum_{j=1}^{r_A} s_j(A) < \infty$, то оператор A называется *ядерным*. Класс всех ядерных операторов будем обозначать буквой $\mathfrak{S}$. Для оператора $A \in \mathfrak{S}$ и произвольного ортонормированного базиса $\{e_\alpha\}$ ряд $\sum_\alpha (Ae_\alpha, e_\alpha)$ абсолютно сходится [1]. Его сумма, которая не зависит от выбора базиса, называется следом оператора A и обозначается символом $\operatorname{sp} A$. Будем говорить, что линейный ограниченный оператор A принадлежит классу $\mathfrak{S}_I$, если его мнимая компонента $A_I = \dfrac{A - A^*}{2i}$ ядерна, и классу $\mathfrak{S}_I^+$, если, кроме того, $A_I \geqq O$.

В первой части настоящей статьи устанавливается формула, связывающая следы мнимых компонент операторов, индуцируемых оператором $A \in \mathfrak{S}_I$ в инвариантных относительно A подпространствах. Существование таких подпространств вытекает из более общих результатов Л. А. Сахновича [2] и В. И. Мацаева [3]. Во второй части вышеупомянутая формула используется для доказательства некоторых теорем об инвариантных подпространствах операторов класса $\mathfrak{S}_I^+$.

В дальнейшем понадобятся следующие теоремы.

1. *Если A — вполне непрерывный линейный оператор, действующий из $\mathfrak{H}$ в $\mathfrak{G}$, то существуют такие ортонормированные последовательности $\{\varphi_j\}_1^{r_A} \subset \mathfrak{H}$ и $\{\psi_j\}_1^{r_A} \subset \mathfrak{G}$, что*

$$Ah = \sum_{j=1}^{r_A} (h, \varphi_j) s_j(A) \psi_j \qquad (h \in \mathfrak{H}). \tag{1}$$

2. *Если $A_1 \in \mathfrak{S}$ и $A_2 \in \mathfrak{S}$, то $\alpha_1 A_1 + \alpha_2 A_2 \in \mathfrak{S}$, где α_1 и α_2 — произвольные комплексные числа.*

3. *Если $A \in \mathfrak{S}$ и B — линейный ограниченный оператор, то $AB \in \mathfrak{S}$ и $BA \in \mathfrak{S}$.*

4. *Если $A \in \mathfrak{S}$, то $A^* \in \mathfrak{S}$.*

5. *Пусть $A \in \mathfrak{S}$ и $P_1, P_2, \ldots$ — последовательность ортопроекторов, которая слабо (следовательно, и сильно) сходится к ортопроектору P_0. Тогда*

$$\mathrm{sp}(P_0 A P_0) = \lim_{m \to \infty} \mathrm{sp}(P_m A P_m). \tag{2}$$

Доказательства утверждений 1—4. приведены в [1]. Формула (2) следует из (1), ибо

$$\mathrm{sp}(P_m A P_m) = \sum_{\alpha=1}^{\infty} (P_m A P_m e_\alpha, e_\alpha) = \sum_{\alpha=1}^{\infty} \sum_{j=1}^{r_A} (P_m e_\alpha, \varphi_j) s_j(A) (P_m \psi_j, e_\alpha) =$$

$$= \sum_{j=1}^{r_A} (P_m \psi_j, P_m \varphi_j) s_j(A) = \sum_{j=1}^{r_A} (P_m \psi_j, \varphi_j) s_j(A) \qquad (m = 0, 1, \ldots)$$

и, следовательно,

$$|\mathrm{sp}(P_0 A P_0) - \mathrm{sp}(P_m A P_m)| \leqq \sum_{j=1}^{r_A} |((P_0 - P_m)\psi_j, \varphi_j)| s_j(A) \leqq$$

$$\leqq \sum_{j=1}^{N} |((P_0 - P_m)\psi_j, \varphi_j)| s_j(A) + \sum_{j=N+1}^{r_A} s_j(A).$$

Мы можем теперь сделать как угодно малым сначала второе слагаемое, выбирая N достаточно большим, а затем первое — за счет выбора m.

1. Пусть $\mathfrak{H}_m$ $(m \in \mathfrak{M})$ — некоторое множество подпространств в $\mathfrak{H}$. Символом $\tilde{\bigcup}_{m \in \mathfrak{M}} \mathfrak{H}_m$ условимся обозначать наименьшее подпространство, содержащее все $\mathfrak{H}_m$. В случае, когда $\mathfrak{M} = \{1, 2\}$, будем писать также $\mathfrak{H}_1 \tilde{\cup} \mathfrak{H}_2$.

Теорема 1. *Если A — ядерный оператор и $\mathfrak{H}^{(1)}, \mathfrak{H}^{(2)}$ — его инвариантные подпространства, то*

$$\mathrm{sp}\, A^{(1)} + \mathrm{sp}\, A^{(2)} = \mathrm{sp}\, A^{\cup} + \mathrm{sp}\, A^{\cap}, \tag{3}$$

где $A^{(1)}, A^{(2)}, A^{\cup}, A^{\cap}$ индуцированы оператором A соответственно в $\mathfrak{H}^{(1)}$, $\mathfrak{H}^{(2)}$, $\mathfrak{H}^{\cup} = \mathfrak{H}^{(1)} \tilde{\cup} \mathfrak{H}^{(2)}$, $\mathfrak{H}^{\cap} = \mathfrak{H}^{(1)} \cap \mathfrak{H}^{(2)}$.

Доказательство. Предположим сначала, что $\mathfrak{H}^{\cap} = 0$. Обозначим через P_2 и P_3 ортопроекторы на $\mathfrak{H}^{(2)}$ и $\mathfrak{H}^{(3)} = \mathfrak{H}^{\cup} \ominus \mathfrak{H}^{(2)}$, действующие в $\mathfrak{H}^{\cup}$, и зададим в $\mathfrak{H}^{(3)}$ оператор $A^{(3)} h = P_3 A^{\cup} h$ $(h \in \mathfrak{H}^{(3)})$. Так как

$$A^{\cup} = P_2 A^{\cup} P_2 + P_3 A^{\cup} P_3 + P_2 A^{\cup} P_3 = A^{(2)} P_2 + A^{(3)} P_3 + P_2 A^{\cup} P_3$$

и, значит, $P_3 A^{\cup} = A^{(3)} P_3$, то $TA^{(1)} = A^{(3)} T$, где T — оператор ортогонального проектирования из $\mathfrak{H}^{(1)}$ на $\mathfrak{H}^{(3)}$. Полагая $B = TA^{(1)}$ и применяя формулу (1), найдем ортонормированные последовательности $\{\varphi_j\}_1^{r_B}$ и $\{\psi_j\}_1^{r_B}$, принадлежащие соответственно подпространствам $\mathfrak{H}^{(1)}$ и $\mathfrak{H}^{(3)}$, для которых

$$Bh = \sum_{j=1}^{r_B} (h, \varphi_j) s_j(B) \psi_j \qquad (h \in \mathfrak{H}^{(1)}).$$

Следовательно,

$$s_j(B)(A^{(1)}\varphi_j, \varphi_j) = \left(\sum_{k=1}^{r_B} (A^{(1)}\varphi_j, \varphi_k) s_k(B) \psi_k, \psi_j \right) = (BA^{(1)}\varphi_j, \psi_j) =$$

$$= (A^{(3)} TA^{(1)}\varphi_j, \psi_j) = (A^{(3)} B\varphi_j, \psi_j) = s_j(B)(A^{(3)}\psi_j, \psi_j),$$

и поэтому

$$(4) \qquad (A^{(1)}\varphi_j, \varphi_j) = (A^{(3)}\psi_j, \psi_j) \qquad (j = 1, 2, \ldots, r_B).$$

Поскольку из равенств $(f, \varphi_j) = 0$ $(f \in \mathfrak{H}^{(1)}, j = 1, 2, \ldots, r_B)$ следует, что $A^{(1)}f = 0$, и, аналогично, из $(g, \psi_j) = 0$ $(g \in \mathfrak{H}^{(3)}, j = 1, 2, \ldots, r_B)$ следует $A^{(3)*} g = 0$, то

$$(5) \qquad \operatorname{sp} A^{(1)} = \sum_{j=1}^{r_B} (A^{(1)}\varphi_j, \varphi_j) = \sum_{j=1}^{r_B} (A^{(3)}\psi_j, \psi_j) = \operatorname{sp} A^{(3)}.$$

Сопоставляя (5) с равенством $\operatorname{sp} A^{\cup} = \operatorname{sp} A^{(2)} + \operatorname{sp} A^{(3)}$, которое вытекает непосредственно из определения следа оператора, получим соотношение

$$(6) \qquad \operatorname{sp} A^{(1)} + \operatorname{sp} A^{(2)} = \operatorname{sp} A^{\cup}.$$

Переходя к общему случаю, обозначим через $P^{(0)}$ ортопроектор на $\mathfrak{H}^{(0)} = \mathfrak{H}^{\cup} \ominus \mathfrak{H}^{\cap}$, действующий в $\mathfrak{H}^{\cup}$, и зададим в $\mathfrak{H}^{(0)}$ оператор $A^{(0)}h = P^{(0)}A^{\cup}h$ $(h \in \mathfrak{H}^{(0)})$. Подпространства $\mathfrak{H}^{(01)} = \mathfrak{H}^{(1)} \ominus \mathfrak{H}^{\cap}$ и $\mathfrak{H}^{(02)} = \mathfrak{H}^{(2)} \ominus \mathfrak{H}^{\cap}$ инвариантны относительно $A^{(0)}$. Через $A^{(01)}$ и $A^{(02)}$ обозначим операторы, индуцированные оператором $A^{(0)}$ соответственно в $\mathfrak{H}^{(01)}$ и $\mathfrak{H}^{(02)}$. В силу уже доказанной части теоремы

$$(7) \qquad \operatorname{sp} A^{(01)} + \operatorname{sp} A^{(02)} = \operatorname{sp} A^{(0)}.$$

Кроме того, очевидно,

$$(8) \qquad \operatorname{sp} A^{(1)} + \operatorname{sp} A^{(2)} = 2 \operatorname{sp} A^{\cap} + \operatorname{sp} A^{(01)} + \operatorname{sp} A^{(02)},$$

$$(9) \qquad \operatorname{sp} A^{\cup} = \operatorname{sp} A^{\cap} + \operatorname{sp} A^{(0)}.$$

Из (7), (8) и (9) следует (3). Теорема доказана.

Легко видеть, что при условиях теоремы 1

$$(10) \qquad \operatorname{sp} A_I^{(1)} + \operatorname{sp} A_I^{(2)} = \operatorname{sp} A_I^{\cup} + \operatorname{sp} A_I^{\cap}.$$

Можно ли утверждать, что формула (10) верна и для произвольных операторов класса $\mathfrak{S}_I$? Мы можем ответить на этот вопрос утвердительно лишь при ограничениях, сформулированных ниже в теоремах 2 и 4.

Теорема 2. *Пусть $\mathfrak{H}^{(1)}$ и $\mathfrak{H}^{(2)}$ — инвариантные подпространства оператора $A \in \mathfrak{S}_I$. Если хотя бы один из операторов $A^{(1)}$, $A^{(2)}$ вполне непрерывен, то имеет место формула* (10).

Доказательство. Повторяя рассуждения, приведенные при доказательстве теоремы 1, и пользуясь введенными там обозначениями, получим для случая $\mathfrak{H}^{(1)} \cap \mathfrak{H}^{(2)} = 0$ следующие равенства:

$$\operatorname{sp} A_I^{(1)} = \sum_{j=1}^{\text{гв}} (A_I^{(1)} \varphi_j, \varphi_j) = \sum_{j=1}^{\text{гв}} (A_I^{(3)} \psi_j, \psi_j) = \operatorname{sp} A_I^{(3)},$$

$$\operatorname{sp} A_I^{\cup} = \operatorname{sp} A_I^{(2)} + \operatorname{sp} A_I^{(3)}.$$

Таким образом,

$$\operatorname{sp} A_I^{(1)} + \operatorname{sp} A_I^{(2)} = \operatorname{sp} A_I^{\cup}. \tag{11}$$

Если же $\mathfrak{H}^{(1)} \cap \mathfrak{H}^{(2)} \neq 0$, то (10) вытекает из соотношений

$$\operatorname{sp} A_I^{(01)} + \operatorname{sp} A_I^{(02)} = \operatorname{sp} A_I^{(0)}, \qquad \operatorname{sp} A_I^{(1)} + \operatorname{sp} A_I^{(2)} = 2 \operatorname{sp} A_I^{\cap} + \operatorname{sp} A_I^{(01)} + \operatorname{sp} A_I^{(02)},$$

$$\operatorname{sp} A_I^{\cup} = \operatorname{sp} A_I^{\cap} + \operatorname{sp} A_I^{(0)}.$$

Формула (11) была получена впервые Г. Э. Кисилевским, предполагавшим, что A — вполне непрерывный оператор класса $\mathfrak{S}_I^{+}$, спектр которого сосредоточен в нуле.

Заметим, что если $\mathfrak{H}^{(1)}$ и $\mathfrak{H}^{(2)}$ — произвольные подпространства в $\mathfrak{H}$ и $\mathfrak{H}^{(1)} = \bigcup_{k=1}^{\infty} \mathfrak{H}_k$ $(\mathfrak{H}_1 \subset \mathfrak{H}_2 \subset \ldots)$, то равенство $\mathfrak{H}^{(1)} \cap \mathfrak{H}^{(2)} = \bigcup_{k=1}^{\infty} (\mathfrak{H}_k \cap \mathfrak{H}^{(2)})$, вообще говоря, не верно. Тем не менее справедливо следующее утверждение.

Теорема 3. *Пусть оператор A имеет инвариантные подпространства $\mathfrak{H}^{(1)}, \mathfrak{H}^{(2)}, \mathfrak{H}_k$ $(\mathfrak{H}_1 \subset \mathfrak{H}_2 \subset \ldots)$, причем $\mathfrak{H}^{(1)} = \bigcup_{k=1}^{\infty} \mathfrak{H}_k$. Если $A \in \mathfrak{S}$, то*

$$\operatorname{sp} A^{\cap} = \lim_{k \to \infty} \operatorname{sp} A_k^{\cap}, \tag{12}$$

где $A^{\cap}$ и $A_k^{\cap}$ индуцированы оператором A соответственно в $\mathfrak{H}^{\cap} = \mathfrak{H}^{(1)} \cap \mathfrak{H}^{(2)}$ и $\mathfrak{H}_k^{\cap} = \mathfrak{H}_k \cap \mathfrak{H}^{(2)}$. Если же $A \in \mathfrak{S}_I$ и все операторы A_k, индуцированные в подпространствах $\mathfrak{H}_k$, вполне непрерывны, то

$$\operatorname{sp} A_I^{\cap} = \lim_{k \to \infty} \operatorname{sp} A_{kI}^{\cap}. \tag{13}$$

Доказательство. Применяя в случае $A \in \mathfrak{S}$ теорему 1 к подпространствам $\mathfrak{H}^{\cap}$ и $\mathfrak{H}_k$, придем к равенству

$$\operatorname{sp} A^{\cap} + \operatorname{sp} A_k = \operatorname{sp} B_k + \operatorname{sp} A_k^{\cap} \tag{14}$$

где B_k — оператор, индуцированный в $\mathfrak{H}^{\cap} \tilde{\cup} \mathfrak{H}_k$. Переходя в (14) к пределу и учитывая, что по формуле (2)

$$\lim_{k \to \infty} \operatorname{sp} A_k = \lim_{k \to \infty} \operatorname{sp} B_k = \operatorname{sp} A^{(1)},$$

получим (12). Равенство (13) следует аналогично из теоремы 2.

Теорема 4. *Пусть* $\mathfrak{H}^{(1)}$ *и* $\mathfrak{H}^{(2)}$ *— инвариантные подпространства оператора* $A \in \mathfrak{S}_I$. *Если* $\mathfrak{H}^{(1)} = \bigcup_{k=1}^{\infty} \mathfrak{H}_k$ $(\mathfrak{H}_1 \subset \mathfrak{H}_2 \subset \ldots)$, *причем все подпространства* $\mathfrak{H}_k$ *инвариантны относительно* A *и операторы* A_k, *индуцированные в* $\mathfrak{H}_k$, *вполне непрерывны, то имеет место формула* (10).

Доказательство. В силу теоремы 2

$$\operatorname{sp} A_{kI} + \operatorname{sp} A_I^{(2)} = \operatorname{sp} A_{kI}^{\cup} + \operatorname{sp} A_{kI}^{\cap}, \tag{15}$$

где $A_k^{\cup}$ и $A_k^{\cap}$ индуцированы соответственно в подпространствах $\mathfrak{H}_k \tilde{\cup} \mathfrak{H}^{(2)}$ и $\mathfrak{H}_k \cap \mathfrak{H}^{(2)}$. Остается, пользуясь формулой (13) и соотношениями

$$\lim_{k\to\infty} \operatorname{sp} A_{kI} = \operatorname{sp} A_I^{(1)}, \qquad \lim_{k\to\infty} \operatorname{sp} A_{kI}^{\cup} = \operatorname{sp} A_I^{\cup},$$

вытекающими из (2), перейти в (15) к пределу.

2. Рассмотрим класс Ω всех линейных ограниченных операторов, действующих в гильбертовом пространстве $\mathfrak{H}$ и имеющих вполне непрерывные мнимые компоненты. Невещественный спектр оператора $A \in \Omega$ представляет собой последовательность $\lambda_j (j=1, 2, \ldots)$, предельные точки которой могут лежать лишь на вещественной оси; корневые подпространства, соответствующие точкам λ_j, конечномерны [4]. Пусть γ_j — положительно ориентированные окружности достаточно малых радиусов, центры которых расположены в точках λ_j, и

$$P_n = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\gamma_j} (A - \lambda E)^{-1}\, d\lambda \qquad (n=1, 2, \ldots). \tag{16}$$

Тогда, согласно известной теореме Ф. Рисса [5], $P_n^2 = P_n$ и подпространства $\mathfrak{H}_n^{(1)} = P_n \mathfrak{H}$, $\mathfrak{H}_n^{(2)} = (E - P_n)\mathfrak{H}$ инвариантны относительно A. При этом $\mathfrak{H}_n^{(1)}$ представляет собой линейную оболочку корневых подпространства, соответствующих точкам $\lambda_1, \lambda_2, \ldots, \lambda_n$, и имеют место соотношения

$$\mathfrak{H} = \mathfrak{H}_n^{(1)} \dot{+} \mathfrak{H}_n^{(2)}, \qquad \mathfrak{H}_1^{(1)} \subset \mathfrak{H}_2^{(1)} \subset \ldots, \mathfrak{H}_1^{(2)} \supset \mathfrak{H}_2^{(2)} \supset \ldots$$

Подпространства

$$\mathfrak{H}^{(n)} = \mathfrak{H}^{(1)}(A) = \tilde{\bigcup_{n=1}^{\infty}} \mathfrak{H}_n^{(1)}, \qquad \mathfrak{H}^{(2)} = \mathfrak{H}^{(2)}(A) = \bigcap_{n=1}^{\infty} \mathfrak{H}_n^{(2)},$$

очевидно, инвариантны относительно A.

Лемма 1. *Если* $\mathfrak{H}_0$ *— инвариантное подпространство оператора* $A \in \Omega$ *и индуцированный в* $\mathfrak{H}_0$ *оператор* A_0 *имеет чисто вещественный спектр, то* $\mathfrak{H}_0 \subset \mathfrak{H}^{(2)}(A)$.

Доказательство. Пользуясь формулой (16) и замечая, что функция

$$(A - \lambda E)^{-1} h_0 = (A_0 - \lambda E)^{-1} h_0 \qquad (h_0 \in \mathfrak{H}_0)$$

голоморфна в невещественных точках, получим равенства $P_n h_0 = 0$ $(n=1, 2, \ldots)$. Следовательно, $\mathfrak{H}_0$ принадлежит всем подпространствам $\mathfrak{H}_n^{(2)}$.

Лемма 2. *Пусть $A \in \Omega$ и $\mathfrak{H}_0$ — инвариантное подпространство оператора A^*. Если оператор A_0^*, индуцированный оператором A^* в $\mathfrak{H}_0$, имеет чисто вещественный спектр, то $\mathfrak{H}_0 \perp \mathfrak{H}^{(1)}(A)$.*

Доказательство. Так как функция

$$((A-\lambda E)^{-1}h, h_0) = (h, (A_0^* - \bar{\lambda}E)^{-1}h_0) \qquad (h \in \mathfrak{H}, \quad h_0 \in \mathfrak{H}_0)$$

голоморфна во всех невещественных точках, то

$$(P_n h, h_0) = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\gamma_j} ((A-\lambda E)^{-1}h, h_0)\, d\lambda = 0$$

и, значит, $\mathfrak{H}_0 \perp \mathfrak{H}_n^{(1)}$ $(n=1, 2, \ldots)$.

Из леммы 1 и 2 легко следует, что $\mathfrak{H}^{(2)}(A) = \mathfrak{H} \ominus \mathfrak{H}^{(1)}(A^*)$.

Теорема 5. *Если A — оператор класса $\mathfrak{S}_I^+$, действующий в пространстве $\mathfrak{H}$, то $\mathfrak{H}^{(1)}(A) \cap \mathfrak{H}^{(2)}(A) = 0$ и $\mathfrak{H}^{(1)}(A) \tilde{\cup} \mathfrak{H}^{(2)}(A) = \mathfrak{H}$. Подпространство $\mathfrak{H}^{(2)}(A)$ является единственным инвариантным, удовлетворяющим указанным выше условиям.*

Доказательство. Пусть снова $\lambda_j (j=1, 2, \ldots)$ — последовательность всех невещественных точек спектра оператора A, γ_j — достаточно малые положительно ориентированные окружности с центрами в точках λ_j,

$$P_n = \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{\gamma_j} (A-\lambda E)^{-1}\, d\lambda, \quad \mathfrak{H}_n^{(1)} = P_n \mathfrak{H}, \quad \mathfrak{H}_n^{(2)} = (E-P_n)\mathfrak{H}.$$

Положим $\mathfrak{H}^{(0)} = \mathfrak{H}^{(1)}(A) \cap \mathfrak{H}^{(2)}(A)$ и обозначим через $A^{(0)}$ оператор, индуцированный в $\mathfrak{H}^{(0)}$. Так как $\mathfrak{H}^{(0)} \subset \mathfrak{H}_n^{(2)}$ и $\mathfrak{H} = \mathfrak{H}_n^{(1)} \dot{+} \mathfrak{H}_n^{(2)}$, то $\mathfrak{H}^{(0)} \cap \mathfrak{H}_n^{(1)} = 0$ $(n=1, 2, \ldots)$. Из теоремы 3 следует, что $\operatorname{sp} A_I^{(0)} = 0$. Поскольку $A_I > 0$, то $Ah_0 = A^*h_0$ $(h_0 \in \mathfrak{H}^{(0)})$, $\mathfrak{H}^{(0)}$ инвариантно относительно A^*, $A^{(0)}$ имеет чисто вещественный спектр и, согласно лемме 2, $\mathfrak{H}^{(0)} \perp \mathfrak{H}^{(1)}(A)$. Этим доказано, что $\mathfrak{H}^{(1)}(A) \cap \mathfrak{H}^{(2)}(A) = 0$.

В силу теоремы 4

$$\operatorname{sp} A_I = \operatorname{sp} A_{nI}^{(1)} + \operatorname{sp} A_{nI}^{(2)}, \tag{17}$$

$$\operatorname{sp} A_I^{\cup} = \operatorname{sp} A_I^{(1)} + \operatorname{sp} A_I^{(2)}, \tag{18}$$

где $A_n^{(1)}, A_n^{(2)}, A^{(1)}, A^{(2)}, A^{\cup}$ индуцированы соответственно в $\mathfrak{H}_n^{(1)}, \mathfrak{H}_n^{(2)}, \mathfrak{H}^{(1)}(A)$, $\mathfrak{H}^{(2)}(A)$ и $\mathfrak{H}^{\cup} = \mathfrak{H}^{(1)}(A) \tilde{\cup} \mathfrak{H}^{(2)}(A)$. Переходя к пределу в (17) и сравнивая результат с (18), получим равенство $\operatorname{sp} A_I = \operatorname{sp} A_I^{\cup}$, которое означает, что $Ah = A^*h$ для каждого вектора $h \in \mathfrak{H} \ominus \mathfrak{H}^{\cup}$. Ввиду леммы 1 $\mathfrak{H} \ominus \mathfrak{H}^{\cup} \subset \mathfrak{H}^{(2)}(A)$, так что $\mathfrak{H}^{(1)}(A) \tilde{\cup} \mathfrak{H}^2(A) = \mathfrak{H}$.

Предположим теперь, что некоторое инвариантное относительно A подпространство $\mathfrak{H}^{(2')}$ удовлетворяет условиям $\mathfrak{H}^{(1)}(A) \cap \mathfrak{H}^{(2')} = 0, \mathfrak{H}^{(1)}(A) \tilde{\cup} \mathfrak{H}^{(2')} = \mathfrak{H}$. Тогда оператор $A^{(2')}$, индуцированный в $\mathfrak{H}^{(2')}$, имеет чисто вещественных спектр и, следовательно, $\mathfrak{H}^{(2')} \subset \mathfrak{H}^{(2)}(A)$. Кроме того,

$$\operatorname{sp} A_I = \operatorname{sp} A_I^{(1)} + \operatorname{sp} A_I^{(2)} = \operatorname{sp} A_I^{(1)} + \operatorname{sp} A_I^{(2')},$$

откуда вытекает, что к подпространству $\mathfrak{H}^{(2)}(A)\ominus\mathfrak{H}^{(2')}$ можно применить лемму 2. Таким образом, $\mathfrak{H}^{(2)}(A)\ominus\mathfrak{H}^{(2')}$ ортогонально к $\mathfrak{H}^{(1)}(A)$ и, значит, $\mathfrak{H}^{(2')}=\mathfrak{H}^{(2)}(A)$.

При помощи теории характеристических функций несамосопряженных операторов близкие к теореме 5 результаты были получены одновременно с автором и независимо от него Ю. П. Гинзбургом. Теорему 5 можно вывести также из некоторых теорем Б. С.-Надя и Ч. Фойяша [6], если потребовать дополнительно, чтобы мнимая компонента оператора A была конечномерной, а характеристическая функция соответствующего оператору A сжатия — внутренней.

Пользуясь теоремой 5 и леммой 1, придем к следующему выводу.

Каждое инвариантное подпространство оператора $A\in\mathfrak{S}_I^+$ есть замыкание линейной оболочки его пересечений с $\mathfrak{H}^{(1)}(A)$ и $\mathfrak{H}^{(2)}(A)$.

Отметим существенность требования диссипативности (т. е. положительности мнимой компоненты) в теореме 5. Пусть $e_0, e_1, e_2, \ldots$ — ортонормированный базис в $\mathfrak{H}$ и $\lambda_1, \lambda_2, \ldots$ — последовательность чисел, для которой $\sum_{j=1}^{\infty}|\lambda_j|<\infty$ $(I_m\lambda_j\neq 0,\ \lambda_j\neq\lambda_k\ (j\neq k))$. Тогда оператор A, определяемый равенствами

$$\begin{aligned} Ae_0 &= \lambda_1 e_1+\lambda_2 e_2+\ldots \\ Ae_1 &= \lambda_1 e_1 \\ Ae_2 &= \lambda_2 e_2 \\ &\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{19}$$

является ядерным, причем замыкание $\mathfrak{H}^{(1)}(A)$ линейной оболочки всех его корневых подпространств, отвечающих невещественным собственным числам, есть ортогональное дополнение к e_0. Вместе с тем не существует, как легко проверить, ненулевого инвариантного подпространства $\mathfrak{H}^{(2)}$, удовлетворяющего условию $\mathfrak{H}^{(1)}(A)\cap\mathfrak{H}^{(2)}=0$. Возможно, однако, что теорема 5 останется в силе, если мы, сохранив диссипативность, заменим условие ядерности другим, менее ограничительным. Пример, аналогичный вышеуказанному, приведен в диссертации И. С. Иохвидова [7].

Теорема 6. *Пусть $A\in\mathfrak{S}_I^+$ и $\mathfrak{H}_0$ — инвариантное подпространство оператора A, принадлежащее $\mathfrak{H}^{(1)}(A)$. Тогда*

$$\mathfrak{H}_0=\tilde{\bigcup_{k=1}^{\infty}}(\mathfrak{H}_0\cap\mathfrak{G}_k), \tag{20}$$

где $\mathfrak{G}_k$ $(k=1,2,\ldots)$ — все корневые подпространства оператора A, соответствующие невещественным точкам спектра.

Доказательство. Обозначим через A_0 и A_0' операторы, индуцированные соответственно в подпространствах $\mathfrak{H}_0$ и $\tilde{\bigcup_{k=1}^{\infty}}(\mathfrak{H}_0\cap\mathfrak{G}_k)$. Так как, очевидно,

$$\tilde{\bigcup_{k=1}^{\infty}}(\mathfrak{H}_0\cap\mathfrak{G}_k)=\mathfrak{H}_0\cap\mathfrak{H}_n^{(1)} \qquad \left(\mathfrak{H}_n^{(1)}=\tilde{\bigcup_{k=1}^{\infty}}\mathfrak{G}_k\right),$$

то

$$\bigcup_{k=1}^{\infty} (\mathfrak{H}_0 \cap \mathfrak{G}_k) = \bigcup_{k=1}^{\infty} (\mathfrak{H}_0 \cap \mathfrak{H}_k^{(1)})$$

и, согласно формулам (2) и (13),

$$\operatorname{sp} A'_{0I} = \lim_{k\to\infty} \operatorname{sp} A'_{kI} = \operatorname{sp} A_{0I},$$

где через A'_k обозначен оператор, индуцированный в $\mathfrak{H}_0 \cap \mathfrak{H}_k^{(1)}$. Таким образом, подпространство $\mathfrak{H}_0 \ominus \bigcup_{k=1}^{\infty} (\mathfrak{H}_0 \cap \mathfrak{G}_k)$ удовлетворяет условию леммы 2 и, значит, ортогонально к $\mathfrak{H}^{(1)}(A)$. Учитывая, что оно вместе с тем принадлежит $\mathfrak{H}^{(1)}(A)$, приходим к формуле (20).

Заметим, что и в теореме 6 нельзя обойтись без требования диссипативности. В самом деле, для оператора, сопряженного с (19), будем иметь

$$A^* e_0 = 0$$

$$A^*(e_0+e_1) = \bar{\lambda}_1 (e_0+e_1)$$

$$A^*(e_0+e_2) = \bar{\lambda}_2 (e_0+e_2)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Здесь корневые подпространства $\mathfrak{G}_k$, соответствующие невещественным собственным числам, одномерны и натянуты на векторы e_0+e_k $(k=1, 2, \ldots)$. При этом $\bigcup_{k=1}^{\infty} \mathfrak{G}_k$ есть все пространство $\mathfrak{H}$, а одномерное подпространство $\mathfrak{H}_0$, натянутое на вектор e_0, не удовлетворяет соотношению (20).

Оператор A, действующий в пространстве $\mathfrak{H}$, называется *вполне несамосопряженным*, если $\mathfrak{H}$ нельзя так представить в виде ортогональной суммы двух нетривиальных инвариантных относительно A подпространств, чтобы оператор, индуцированный в одном из них, был самосопряженным. Для любого оператора A пространство $\mathfrak{H}$ однозначно представимо в виде ортогональной суммы подпространств $\mathfrak{H}_0(A)$ и $\mathfrak{H}^{\perp}(A)$, удовлетворяющих следующим условиям: 1. $\mathfrak{H}_0(A)$ и $\mathfrak{H}^{\perp}(A)$ инвариантны относительно A, 2. в $\mathfrak{H}_0(A)$ индуцируется вполне несамосопряженный, а в $\mathfrak{H}^{\perp}(A)$ – самосопряженный оператор. Если в некотором инвариантном относительно A и A^* подпространстве индуцируется самосопряженный оператор, то оно принадлежит $\mathfrak{H}^{\perp}(A)$ [4].

Теорема 7. *Пусть $A \in \mathfrak{S}_I^+$, $\lambda_1, \lambda_2, \ldots$ – последовательность всех невещественных собственных чисел оператора A и ν_k – размерность корневого подпространства $\mathfrak{G}_k$, соответствующего числу λ_k. Тогда*

$$\sum_{k=1}^{\infty} \nu_k \operatorname{Im} \mu_k \leqq \operatorname{sp} A_I, \tag{21}$$

причем знак равенства имеет место в том и только том случае, когда $\mathfrak{H}^{\perp}(A) = \mathfrak{H}^{(2)}(A)$.

Доказательство. Согласно теоремам 5 и 4

$$\mathrm{sp}\, A_I^{(1)} + \mathrm{sp}\, A_I^{(2)} = \mathrm{sp}\, A_I, \tag{22}$$

где $A^{(1)}$ и $A^{(2)}$ индуцированы соответственно в $\mathfrak{H}^{(1)}(A)$ и $\mathfrak{H}^{(2)}(A)$. Легко видеть, что след мнимой компоненты оператора, индуцированного в $\mathfrak{G}_k$, равен $\nu_k \operatorname{Im} \lambda_k$. Применяя теорему 2 к подпространству $\mathfrak{H}_n^{(1)} = \bigcup_{k=1}^{n} \mathfrak{G}_k$ и пользуясь формулой (2), получим:

$$\mathrm{sp}\, A_I^{(1)} = \sum_{k=1}^{\infty} \nu_k \operatorname{Im} \lambda_k. \tag{23}$$

Из (22) и (23) следует неравенство (21). В нем тогда и только тогда будет достигаться знак равенства, когда $\mathrm{sp}\, A_I^{(2)}$. Если последнее соотношение выполняется, то $Ah = A^*h$ $(h \in \mathfrak{H}^{(2)}(A))$ и, следовательно, $\mathfrak{H}^{(2)}(A) \subset \mathfrak{H}^{\perp}(A)$. С другой стороны, согласно лемме 1, $\mathfrak{H}^{\perp}(A) \subset \mathfrak{H}^{(2)}(A)$. Обратно, если $\mathfrak{H}^{\perp}(A) = \mathfrak{H}^{(2)}(A)$, то, очевидно, $\mathrm{sp}\, A_I^{(2)} = 0$.

Из теоремы 7 вытекает следующее утверждение, доказательство которого впервые было получено М. С. Лившицем [8] и впоследствии упрощено Б. Р. Мукминовым [9].

Теорема 8. *Если A — вполне несамосопрженный оператор класса $\mathfrak{S}_I^+$, действующий в $\mathfrak{H}$, то имеет место неравенство* (21), *причем знак равенства достигается в том и только том случае, когда $\mathfrak{H}^{(1)}(A) = \mathfrak{H}$.*

Цитированная литература

[1] Ц. Гохберг и М. Г. Крейн, *Введение в теорию линейных несамосопряженных операторов* (Москва, 1965).

[2] Л. А. Сахнович, Исследование „треугольной модели" несамосопряженных операторов, *Изв. высших учебных заведений, Математика*, **4 (11)** (1959), 141—149.

[3] В. И. Мацаев, Об одном классе вполне непрерывных операторов, *ДАН СССР*, **139** (1961), 548—552.

[4] М. С. Бродский и М. С. Лившиц, Спектральный анализ несамосопряженных операторов и промежуточные системы, *УМН*, **13:1** (79), (1958), 3—85.

[5] Ф. Рисс и Б. С.-Надь, *Лекции по функциональному анализу* (Москва, 1954).

[6] B. Sz.-Nagy et C. Foiaş, Sur les contractions de l'espace de Hilbert. VII, VIII, *Acta Sci. Math.*, **25** (1964), 12—36, 38—70.

[7] И. С. Иохвидов, *Самосопряженные и унитарные операторы в пространствах с индефинитной метрикой*, Диссертация, Одесса (1950).

[8] М. С. Лившиц, О спектральном разложении линейных несамосопряженных операторов, *Матем. сб.*, **34 (76)** (1954), 145—199.

[9] Б. Р. Мукминов, О разложении по собственным функциям диссипативных ядер, *ДАН СССР*, **99** (1954), 499—502.

(Поступило 16. XII. 1965)

Characterization of some classes of measures

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I. Introduction

Let $M(G)$ be the set of (bounded regular Borel) measures μ on a locally compact abelian group G. The following theorem is a well-known characterization of those measures which are absolutely continuous (with respect to the Haar measure of G), given in terms of the translates μ_x (where $\mu_x(E)=\mu(E-x)$).

Theorem A. *Let $\mu \in M(G)$. Then μ is absolutely continuous if and only if $\|\mu_x-\mu\| \to 0$ as $x \to 0$.*

For a proof see ([5], p. 230). The norm in the statement of Theorem A is the usual measure norm ($=$total variation).

In this paper we introduce two other norms for $M(G)$. Using them we give 1) a characterization of the measures in $M(G)$ whose Fourier—Stieltjes transforms vanish at infinity, and 2) a characterization of the continuous ($=$non-atomic) measures in $M(G)$. In each case the necessary and sufficient condition is similar to that in Theorem A — namely, that as $x \to 0$, μ_x must approach μ in a suitable norm. In case 2) we must restrict ourselves to metrizable groups.

II. Characterization of measures whose Fourier—Stieltjes transforms vanish at infinity

Let Γ denote the character group of G. If $\mu \in M(G)$ let

$$\|\mu\|_\Gamma = \sup_{\gamma \in \Gamma} |\hat{\mu}(\gamma)|$$

where $\hat{\mu}$ is the Fourier—Stieltjes transform of μ — that is $\hat{\mu}(\gamma)=\int_G \overline{(\gamma, t)}\,d\mu(t)$. Since μ is determined by the values of $\hat{\mu}$ on Γ [5], we have $\|\mu\|_\Gamma=0$ if and only if μ is the 0 measure. The other conditions that $\|\cdot\|_\Gamma$ be a norm are readily verified. Here is our characterization.

Theorem B. *Let $\mu \in M(G)$. Then $\hat{\mu}$ vanishes at infinity if and only if $\|\mu_x-\mu\|_\Gamma \to 0$ as $x \to 0$.*

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[1]) Research supported by the National Science Foundation Grant GP-3930.

Proof. Suppose first that $\hat{\mu}$ vanishes at infinity. Since

$$\hat{\mu}_x(\gamma) = \int_G \overline{(\gamma, t)}\, d\mu_x(t) = \int_G \overline{(\gamma, t)}\, d\mu(t-x) = \int_G \overline{(\gamma, t+x)}\, d\mu(t) = \overline{(\gamma, x)}\,\hat{\mu}(\gamma),$$

we have

$$\|\mu_x - \mu\|_\Gamma = \sup_{\gamma\in\Gamma} |\hat{\mu}_x(\gamma) - \hat{\mu}(\gamma)| = \sup_{\gamma\in\Gamma} |(\gamma, x) - 1|\cdot|\hat{\mu}(\gamma)|. \tag{1}$$

Since, by assumption, $\hat{\mu}$ vanishes at infinity, given $\varepsilon > 0$ there exists a compact $K \subset \Gamma$ such that $|\hat{\mu}(\gamma)| < \frac{\varepsilon}{2}$ if $\gamma \in \Gamma - K$. Moreover, the set U of all x in G such that $|(\gamma, x) - 1| < \varepsilon/\|\mu\|_\Gamma$ for all $\gamma \in K$ is a neighborhood of 0 in G. If $\gamma \in K$ we thus have

$$|(\gamma, x) - 1|\cdot|\hat{\mu}(\gamma)| < (\varepsilon/\|\mu\|_\Gamma)\cdot\|\mu\|_\Gamma = \varepsilon \quad (x \in U), \tag{2}$$

while if $\gamma \in \Gamma - K$ we have

$$|(\gamma, x) - 1|\cdot|\hat{\mu}(\gamma)| < 2\cdot\frac{\varepsilon}{2} = \varepsilon \qquad (x \in G). \tag{3}$$

From (1), (2), (3) it follows that $\|\mu_x - \mu\|_\Gamma < \varepsilon$ if $x \in U$. That is $\|\mu_x - \mu\| \to 0$ as $x \to 0$. This proves half the theorem.

To prove the other half we need a lemma (see [1]).

Lemma. *Let U be any neighborhood of 0 in the locally compact abelian group G. Then there exists a compact subset K of Γ (the character group of G) such that for any $\gamma \in \Gamma - K$ there exists $x \in U$ with* $\operatorname{Re}(\gamma, x) \leqq 0$.

Now suppose $\|\mu_x - \mu\|_\Gamma \to 0$ as $x \to 0$. We must show that $\hat{\mu}$ vanishes at infinity. Given $\varepsilon > 0$ choose a neighborhood U of 0 in G such that $\|\mu_x - \mu\|_\Gamma < \varepsilon$ $(x \in U)$. Then

$$|(\gamma, x) - 1|\cdot|\hat{\mu}(\gamma)| < \varepsilon \quad (\gamma \in \Gamma;\ x \in U). \tag{4}$$

For this U choose $K \subset \Gamma$ according to the lemma. Then if $\gamma \in \Gamma - K$ there exists $x \in U$ with $\operatorname{Re}(\gamma, x) \leqq 0$ so that $|(\gamma, x) - 1| > 1$. Using this x in (4) we have $|\hat{\mu}(\gamma)| < \varepsilon$ if $\gamma \in \Gamma - K$. This completes the proof.

III. Characterization of continuous measures

If G is a non-discrete metrizable group then its character group Γ is σ-compact. In Γ there is a sequence of open subsets A_n with compact closure satisfying $A_1 \subset A_2 \subset \ldots$, $\lim_{n\to\infty} m(A_n) = \infty$, and such that

$$M(f) = \lim_{n\to\infty} \frac{1}{m(A_n)} \int_{A_n} f(\gamma)\, d\gamma \tag{5}$$

exists for all almost periodic functions f on Γ and is equal to the mean value of f ([2]). Here $m(A_n)$ means the Haar measure of A_n. Hewitt and Stromberg ([3]) have shown that the limit in (5) will exist for many other functions as well, and, in fact they proved

Lemma. *Let $\mu \in M(G)$. Then μ is a continuous measure if and only if $M(|\hat{\mu}|)=0$.*

We now define our second norm. If $\mu \in M(G)$ where G is metrizable, let

$$\mathcal{N}(\mu) = \sup_n \frac{1}{m(A_n)} \int_{A_n} |\hat{\mu}(\gamma)|\, d\gamma.$$

Our second main result is

Theorem C. *Let G be a metrizable locally compact abelian group and let $\mu \in M(G)$. Then μ is a continuous measure if and only if $\mathcal{N}(\mu_x - \mu) \to 0$ as $x \to 0$.*

Proof. First suppose that μ is continuous. Then by the lemma we have $M(|\hat{\mu}|) = \lim_{n\to\infty} \frac{1}{m(A_n)} \int_{A_n} |\hat{\mu}(\gamma)| d\gamma = 0$. Hence, given $\varepsilon > 0$ there exists N such that

$$\frac{1}{m(A_n)} \int_{A_n} |\hat{\mu}(\gamma)|\, d\gamma < \varepsilon/2 \quad (n \geqq N). \tag{6}$$

Moreover, the set U of x in G such that $|(\gamma, x) - 1| < \varepsilon/\|\mu\|_\Gamma$ for all $\gamma \in \overline{A_N}$ is a neighborhood of 0 in G. Thus, if $n \geqq N$ we have from (6)

$$\frac{1}{m(A_n)} \int_{A_n} |(\gamma, x) - 1| \cdot |\hat{\mu}(\gamma)|\, d\gamma \leqq \frac{1}{m(A_n)} \int_{A_n} 2|\hat{\mu}(\gamma)| < \varepsilon \qquad (x \in G).$$

Also, if $n \leqq N$ then $A_n \subseteqq \overline{A_N}$ and so, if $x \in U$

$$\frac{1}{m(A_n)} \int_{A_n} |(\gamma, x) - 1| \cdot |\hat{\mu}(\gamma)|\, d\gamma \leqq \|\mu\|_\Gamma \cdot \frac{1}{m(A_n)} \int_{A_n} |(\gamma, x) - 1|\, d\gamma \leqq$$

$$\leqq \|\mu\|_\Gamma (\varepsilon/\|\mu\|_\Gamma) \cdot \frac{1}{m(A_n)} \int_{A_n} d\gamma = \varepsilon.$$

Thus, if $x \in U$,

$$\sup_n \frac{1}{m(A_n)} \int_{A_n} |(\gamma, x) - 1| \cdot |\hat{\mu}(\gamma)|\, d\gamma \leqq \varepsilon,$$

or

$$\sup_n \frac{1}{m(A_n)} \int_{A_n} |\hat{\mu}_x(\gamma) - \hat{\mu}(\gamma)|\, d\gamma \leqq \varepsilon,$$

or

$$\mathcal{N}(\mu_x - \mu) \leqq \varepsilon$$

Thus, $\mathcal{N}(\mu_x - \mu) \to 0$ as $x \to 0$. This proves half the theorem.

Now suppose that $\mathcal{N}(\mu_x - \mu) \to 0$ as $x \to 0$. We must prove that μ is continuous. Given $\varepsilon > 0$ choose a symmetric neighborhood U of 0 in G such that

$$\mathcal{N}(\mu_x - \mu) < \varepsilon \qquad (x \in U). \tag{7}$$

Let φ be the function on G defined by

$$\varphi(t)=1/m(U) \qquad (t\in U),$$
$$\varphi(t)=0 \qquad (t\in G-U).$$

(We are now denoting the Haar measure on G, as well as that on Γ, by m. We may clearly assume that $m(U)<\infty$.) Then $\hat{\varphi}$, the Fourier transform of φ, is real-valued (since U is symmetric) and $\hat{\varphi}$ vanishes at infinity by the Riemann—Lebesgue theorem. Thus, for some compact $K\subset\Gamma$, $\hat{\varphi}(\gamma)\leqq\frac{1}{2}$ if $\gamma\in\Gamma-K$. That is,

$$\hat{\varphi}(\gamma)=\int_G\overline{(\gamma,x)}\varphi(x)\,dx=\frac{1}{m(U)}\int_U(\gamma,x)\,dx\leqq\frac{1}{2} \qquad (\gamma\in\Gamma-K).$$

Hence

$$\frac{1}{m(U)}\int_U[1-(\gamma,x)]\,dx\geqq\frac{1}{2} \qquad (\gamma\in\Gamma-K),$$

and so

$$\frac{1}{m(U)}\int_U|(\gamma,x)-1|\,dx\geqq\frac{1}{2} \qquad (\gamma\in\Gamma-K). \tag{8}$$

Now from (7) we have for any n

$$\frac{1}{m(A_n)}\int_{A_n}|(\gamma,x)-1|\cdot|\hat{\mu}(\gamma)|\,d\gamma<\varepsilon \qquad (x\in U).$$

If we multiply by $\dfrac{1}{m(U)}$, integrate over U, and invert the order of integration we obtain

$$\frac{1}{m(A_n)}\int_{A_n}|\hat{\mu}(\gamma)|\,d\gamma\cdot\frac{1}{m(U)}\int_{A_n}|(\gamma,x)-1|\,dx<\varepsilon.$$

Then certainly

$$\frac{1}{m(A_n)}\int_{A_n-K}|\hat{\mu}(\gamma)|\,d\gamma\cdot\frac{1}{m(U)}\int_U|(\gamma,x)-1|\,dx<\varepsilon.$$

But if $\gamma\in A_n-K$ then, by (8), $\dfrac{1}{m(U)}\displaystyle\int_U|(\gamma,x)-1|dx\geqq\frac{1}{2}$. Hence

$$\frac{1}{m(A_n)}\int_{A_n-K}|\hat{\mu}(\gamma)|\,d\gamma<2\varepsilon.$$

Moreover, it is certainly true for large n that

$$\frac{1}{m(A_n)}\int_K|\hat{\mu}(\gamma)|\,d\gamma<\varepsilon$$

since $m(A_n)\to\infty$ as $n\to\infty$. Hence

$$\frac{1}{m(A_n)}\int\limits_{A_n}|\hat{\mu}(\gamma)|\,d\gamma<3\varepsilon$$

for large n, which proves that $M(|\hat{\mu}|)=\lim\limits_{n\to\infty}\frac{1}{m(A_n)}\int\limits_{A_n}|\hat{\mu}(\gamma)|d\gamma=0$. By the lemma, μ is continuous and the proof is complete.

IV. Remark on another class of measures

We have made an attempt at another result along the lines of Theorems A, B, C. Let Δ be the maximal ideal space of the measure algebra $M(G)$. That is, Δ is the space of continuous complex-valued homomorphisms h on $M(G)$. If for $\mu\in M(G)$ we define $\|\mu\|_\Delta=\sup\limits_{h\in\Delta}|h(\mu)|$ then, since $h(\mu)=\hat{\mu}(h)$ where $\hat{\mu}$ is the Gelfand transform of μ, $\|\mu\|_\Delta$ is the spectral norm of μ [4: p. 76]. It is now natural to ask: For what μ is it true that $\|\mu_x-\mu\|_\Delta\to0$ as $x\to0$?

For each $x\in G$ let σ_x be the unit mass concentrated at x. For $h\in\Delta$ the function χ_h defined by

$$\chi_h(x)=h(\sigma_x)\qquad(x\in G)$$

is easily seen to be a group character of G. However, χ_h need not be continuous. If we could answer positively a certain question about these χ_h we could give a characterization of the kernel of the hull of $L^1(G)$ in $M(G)$ — the set of all $\mu\in M(G)$ such that $\hat{\mu}(h)=0$ for all $h\in\Delta-\hat{G}$. The question whose answer we are unable to establish is this: Are the h for which χ_h is discontinuous dense in $\Delta-\hat{G}$? If the answer to this question is yes then we can easily establish the following:

Let $\mu\in M(G)$. Then μ is in the kernel of the hull of $L^1(G)$ if and only if $\|\mu_x-\mu\|_\Delta\to0$ as $x\to0$.

References

[1] R. R. Goldberg and A. B. Simon, The Riemann—Lebesgue theorem on groups, *Acta Sci. Math.*, **27** (1966), 35—40.
[2] E. Hewitt and K. Ross, *Abstract Harmonic Analysis* (Berlin, 1963).
[3] E. Hewitt and K. R. Stromberg, A remark on Fourier—Stieltjes transforms, *An. Acad. Brasil. Ci.*, **34** (1962), 175—180.
[4] L. Loomis, *Introduction to Abstract Harmonic Analysis* (New York, 1953).
[5] W. Rudin, Measure algebras on abelian groups, *Bull. Amer. Math. Soc.*, **65** (1959), 227—247.

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(Received January 11, 1966)

A type of extension of Banach spaces

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The purpose of this article is the study of a certain relation between a Banach space X and a linear subspace Y. A subset F of the unit ball of X^* is called a *Y-boundary* if for each y in Y, $\|y\| = \sup\{|f(y)| : f \in F\}$; X is a *bound extension* of Y if every Y-boundary is an X-boundary. By requiring that X itself admit no bound extension one has an obverse to the "projective resolution" of a compact Hausdorff space defined and constructed by GLEASON [1; Theorem 3. 2]. We believe, moreover, that the theory of P_1 spaces is naturally included in the present discussion.

Banach spaces may be real or complex; but since certain arguments are more difficult in the complex case, this case is sometimes treated, to the exclusion of the other.

Lemma 0. *X is a bound extension of its subspace Y if and only if there is no semi-norm $\|x\|_1$ on X except $\|x\|$ itself, such that $\|x\|_1 \leqq \|x\|$ for all x, and $\|y\|_1 = \|y\|$ for all y.*

Proof. If F is a Y-boundary in the unit ball of X^*, we can define $\|x\|_1 \equiv \equiv \sup\{|f(x)| : f \in F\}$. Then $\|x\|_1 \equiv \|x\|$ if and only if F is an X-boundary, yielding the direct implication. Conversely, if $\|x\|_1$ is a semi-norm as given in Lemma 0., we can take $F = \{f \in X^* : |f(x)| \leqq \|x\|_1$ for all $x\}$ as a Y-boundary; if F is an X-boundary $\|x\|_1 \equiv \|x\|$.

We shall use the term *P_1 space* for a Banach space Z with the "extension property": for each Banach space X, subspace Y, and bounded linear transformation T of Y into Z, there is an extension T' of T mapping X into Z, and $\|T'\| = \|T\|$. For a discussion of P_1 spaces see KELLEY [3]. A natural and obvious example is the space $B(S)$ of all bounded function on an abstract set S, under the usual sup-norm; this example shows that every Banach space can be extended to a P_1 space. The best possible extension is discussed in the next two paragraphs.

Theorem 1. *If X is a P_1 space and Y a linear subspace of X, there is a subspace $Z \supseteqq Y$, which is a P_1 space and a bound extension of Y.*

Proof. Consider the family N of semi-norms on X subject to the conditions of Lemma 0. Zorn's Lemma yields the existence of a minimal element $\|x\|_0$ in N. X being a P_1 space there exists a linear transformation T of X into itself such that $T(y) = y$ for each y in Y and $\|T(x)\| \leqq \|x\|_0$ for each x in X. Then the semi-norm $\|x\|_1 = \|T(x)\|$, $x \in X$, belongs to N, whence $\|x\|_1$ coincides with $\|x\|_0$. Still another element of N is defined by $\|x\|_2 \equiv \limsup\limits_n \left\| \frac{1}{n} \sum\limits_1^n T^i(x) \right\|$. (In fact the limit exists.)

Since $\|x\|_2 \leqq \|x\|_0$ for all x, and T fixes each member of Y, $\|x\|_2 \equiv \|x\|_0$. T being a contraction of X (with respect to its original norm), it is well known that for every x, $\|(I-T)x\|_2 = 0$, or what is the same, $T^2 = T$. The space $Z = T(X)$ contains Y and is clearly a P_1 space. Finally, observe that each semi-norm $\|z\|_1'$ on Z has a natural extension to X by the definition $\|T(x)\|_1 \equiv \|x\|_1$; the minimal nature of $\|T(x)\|$ shows then that Z, with its given norm, is a bound extension of Y.

Corollary 2. *Let Z be a P_1 space which is a bound extension of a subspace Y, and U an isometry of Y into a space X such that X is a bound extension of $U(Y)$. There is a unique linear transformation V of X into Z for which*

(i) $UV(y) = y$ *for* $y \in Y$, (ii) $\|V\| \leqq 1$.

Moreover, V is an isometry.

Proof. The existence of V is assured by the hypotheses that U be isometric and Z a P_1 space. Lemma 0 shows that V is in fact an isometry of X into Z; let V_1 be another transformation with the required properties. Then there is a linear contraction S of Z such that $SV_1(x) \equiv SV(x)$; by an argument similar to that in Theorem 1, for each Z it is true that $\|z\| = \limsup \left\| \frac{1}{n} \sum_1^n S^i(z) \right\|$; then $S = 1$ and $V_1 = V$.

In the two lemmas and theorem immediately below A is a compact Hausdorff space, Y a linear subspace of $C(A)$ (real or complex), and by hypothesis there is no proper closed subset of A in which every member of Y assumes its maximum modulus. Lemmas 3 and 4 contain the irreducible kernel of analysis necessary for "concrete" applications.

Lemma 3. *If U is a non-empty open subset of A and $\frac{\pi}{4} > \varepsilon > 0$, there is an element y in Y which assumes its maximum modulus only in*

$$U \cap \{a\colon \operatorname{Re} y(a) > \|y\| \cos \varepsilon\}.$$

Proof. Let y_1 be an element of Y which assumes its maximum modulus only in U and moreover $\|y\| = \max \operatorname{Re} y_1$. Let y_2 be an element of Y which attains its maximum modulus only in the set

$$U \cap \left\{a\colon \operatorname{Re} y_1(a) > \|y\|_1 \cos \frac{\varepsilon}{3}\right\}, \quad \text{and} \quad \|y\|_2 = \max \operatorname{Re} y_2.$$

We shall show that for all sufficiently large n the functions $h_n = ny_2 + y_1$ in Y are suitable for the present lemma.

Indeed suppose $a_n \in A$ and $|h_n(a_n)| = \|h_n\|$, $n = 1, 2, 3, \ldots$. Since $\|h_n\| \geqq n\|\|y_2\| + \|y\|_1 \cos \frac{\varepsilon}{3}$ for each n, $\left(n\|y\|_2\| + \|y_1\| \cos \frac{\varepsilon}{3}\right)^2 \leqq |ny_2(a_n)|^2 + 2n \operatorname{Re}[y_2(a_n)\overline{y_1(a_n)}] + |y_1(a_n)|^2$. Thus $|y_2(a_n)| \to \|y_2\|$ and $\liminf \operatorname{Re} [y_2(a_n)\overline{y_1(a_n)}] \geqq \|y_1\| y_2\| \cdot \cos \frac{\varepsilon}{3}$.

But the argument of $y_1(a_n)$ is ultimately confined to $\left[-\frac{\varepsilon}{3}, \frac{\varepsilon}{3}\right]$ so that the argument of $y_2(a_n)$ is ultimately confined to $\left[-\frac{2\varepsilon}{3}, \frac{2\varepsilon}{3}\right]$; the argument of $h_n(a_n)$ is then also restricted as asserted, for n sufficiently large.

Lemma 4. *Let $f \in C(A)$ and $0 \leqq d < \|f\|$. Then for some y in Y, $\|y+f\| < \|y\| - d$.*

Proof. We can assume that f attains the value $\|f\|$ in A. Let $d < s < \|f\|$ and V be a neighborhood of -1 in the plane. Then there is a function y in Y which attains its maximum modulus, 1, only in the set $\{a\colon \operatorname{Re} f(a) > s\} \cap \{a\colon y(a) \in V\}$. Once y is chosen we can estimate $\|f+ny\|$ as $n \to \infty$: $|f(a)+ny(a)|^2 = |f(a)|^2 + 2n \operatorname{Re}[f(a)\overline{y(a)}] + n^2 |y(a)|^2 \leqq n^2 + 2nB + o(n)$, where $B = \sup\{\operatorname{Re}[\bar{\lambda} f(a)]\colon \lambda \in V, a \in A, \operatorname{Re} f(a) \geqq s\}$. If the lemma is false then for every choice of s and V we must have $\|f+ny\| \geqq n-d$, whence $B \geqq -d$. Passing to the limit as V contracts to -1, we obtain an a such that $\operatorname{Re} f(a) \geqq s$, and $-\operatorname{Re} f(a) \geqq -d$, or $\operatorname{Re} f(a) \leqq s$, a contradiction proving the lemma.

Theorem 5. *$C(A)$ is a bound extension of Y.*

Proof. If $\|x\|_1$ is a semi-norm as in Lemma 0, then for every $y \in Y, f \in C(A)$, we have $\|y+f\| \geqq \|y+f\|_0 \geqq \|y\| - \|f\|_0$. By Lemma 4, $\|f\| = \|f\|_0$.

Corollary 6. *A Banach space X is a bound extension of a subspace Y if and only if the unit ball of X^* contains a w^*-closed X-boundary F which is a minimal w^*-closed Y boundary.*

Proof. The converse assertion is clear inasmuch as any Y-boundary is an X-boundary. On the other hand if the set F exists we can consider that $Y \subseteqq X \subseteqq C(F)$ and by Theorem 5, $C(F)$ is a bound extension of Y. An easy application of the Hahn—Banach Theorem shows that X, too, is a bound extension of Y.

Let us apply the previous remarks to a P_1 space X, and a minimal w^*-closed X-boundary A in the unit ball of X^*. By Theorem 5, $C(A)$ is a bound extension of the P_1 space X, whence $X = C(A)$. Again from the definition of P_1 space, there is a projection T of the Banach space of bounded functions on A onto $C(A)$, the projection T having norm one. Since $C(A)$ contains the constant functions, it is plain that T must preserve the class of non-negative real functions. In particular if h is the characteristic function of an open subset U of A, then $T(h) = 1$ on U and $T(h) = 0$ on the complement of U^-. Then U^- must be open: A is extremally disconnected (KELLEY [3]).

To complete our previous considerations, and obtain incidentally a converse to the last remark, we require a lemma on regular open sets. For the necessary theory of Boolean algebras, one may consult HALMOS [2], in particular pages 13—17. We adopt the notation that ϱS be the interior of the closure of S for any subset S in a topological space, and $\mathscr{R}(M)$ be the Boolean algebra of regular open subsets of M.

Lemma 7. *Let f be a continuous mapping of a compact Hausdorff space M onto a Hausdorff space N, such that $f(S) \neq N$ for any proper closed subset S of M. There is defined a Boolean isomorphism m of $\mathscr{R}(M)$ onto $\mathscr{R}(N)$:*

$$mU = \varrho f(U), \qquad U \in \mathscr{R}(M).$$

The inverse s is given by

$$sV = \varrho f^{-1}(V), \qquad V \in \mathscr{R}(N).$$

Proof. We verify first that m and s are inverse to each other. If $V \in \mathscr{R}(N)$, surely $V \subseteqq msV$; since $f^{-1}(V)$ is dense in $f^{-1}(msV)$, V is dense in msV and $V = msV$. If $U \in \mathscr{R}(M)$, then $f(smU)$ contains the interior of $f(U)$, so $f(U') \cup f(smU)$ is dense in N and $U' \cup smU$ is dense in M, yielding $U \subseteqq smU$. From the fact that $f(U)$ is dense in $f(smU)$ it follows similarly that $U \supseteqq smU$.

The identity $\varrho(E \cup F) = \varrho E \vee \varrho F$ for arbitrary subsets $E, F \subseteqq N$, shows that $m(U_1 \vee U_2) = mU_1 \vee mU_2$ for $U_1, U_2 \in \mathscr{R}(M)$; this depends on the fact that $U_1 \cup U_2$ is dense in $U_1 \vee U_2$. To see that $m(U_1 \cap U_2) = mU_1 \cap mU_2$, observe that for certain open subsets W_1 and W_2 of N, $f^{-1}(W_i) \subseteqq U_i$ and $f^{-1}(W_i)$ is dense in U_i, $i = 1, 2$. Then $f^{-1}(W_1 \cap W_2)$ is dense in $U_1 \cap U_2$, and $m(U_1 \cap U_2) = \varrho(W_1 \cap W_2) = \varrho W_1 \cap \varrho W_2 = mU_1 \cap mU_2$ (lemma 4, p. 15, [2]). The facts now established for m and s complete the proof.

Corollary 8. *If N is extremally disconnected, f is a homeomorphism* (GLEASON [1; Lemma 2.3]).

Proof. Since $\mathscr{R}(N)$ contains only closed subsets of N, $\mathscr{R}(M)$ contains only subsets of the form $f^{-1}(W)$ for a subset W open in N; the same form prevails for all open subsets of M, whence f is one-to-one.

Returning to the general problem, we begin with a Banach space X and a minimal w^*-closed X-boundary F in the unit ball of X^*. Also, let Z be a bound P_1 extension of X; we know at the outset that Z is isometric with $C(A)$ for some extremally disconnected compact Hausdorff space A. We shall show that A is the Stone space of the Boolean algebra $\mathscr{R}(F)$ and that $\mathscr{R}(F)$ is independent (to within isomorphism) of the choice of F.

The first step is to consider a w^*-closed subset F_1 of the unit ball of Z^* whose restriction to X is exactly F; since Z is a bound extension of X, F_1 is a Z-boundary. Since F is a minimal closed X-boundary, we can suppose that F_1 is a minimal closed Z-boundary. If we use the familiar representation of Z^* by countably additive Borel measures in A, we see that F_1 must contain, for each $a \in A$, a measure with mass 1 at $\{a\}$; this does not depend on the disconnectedness of A. It is convenient to write $\lambda \cdot a$ for the functional $f \to \lambda f(a)$, $f \in C(A)$, $a \in A$, λ a complex number. The measures in F_1 which can be represented in the manner just described form a closed subset which is a boundary for Z and consequently they exhaust F_1, by the minimality. The mapping π of F_1 onto A given by $\pi(\lambda \cdot a) = a$ for $\lambda \cdot a$ in F_1, is continuous and fulfills the conditions of Lemma 7 and Corollary 8, insuring that π is a homeomorphism of F_1 onto A. The Boolean algebra $\mathscr{R}(F_1)$ (or $\mathscr{R}(A)$) determines A to within homeomorphism, while $\mathscr{R}(F_1) \cong \mathscr{R}(F)$ by Lemma 7. This is the conclusion sought, in view of the fact that $\mathscr{R}(A)$ coincides with the Boolean algebra of open-closed subsets of A.

References

[1] A. M. GLEASON, Projective topological spaces, *Illinois J. Math.*, **2** (1958), 482—489.
[2] P. R. HALMOS, *Lectures on Boolean Algebras* (Princeton, 1963).
[3] J. L. KELLEY, Banach spaces with the extension property, *Trans. Amer. Math Soc.*, **72** (1952), 323—326.

(Received December 23, 1965)

On interpolation functions

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If X is a locally compact space provided with a positive measure μ, we denote by L^p_ζ, where $1<p<\infty$ and ζ is a positive μ-measurable function, the space of μ-measurable functions a such that ζa is of μ-integrable pth power. We provide L^p_ζ with the norm

$$\|a\|_{L^p_\zeta} = \Big(\int_X |\zeta a|^p \, d\mu\Big)^{1/p}.$$

A function $H=H(z_0, z_1)$, defined, Borel measurable, and positive for $z_0>0$, $z_1>0$, is said to be an *interpolation function* of power p if and only if whenever π is a linear mapping which is continuous from $L^p_{\zeta_0}$ into $L^p_{\zeta_0}$ and from $L^p_{\zeta_1}$ into $L^p_{\zeta_1}$ then π is also continuous from $L^p_{H(\zeta_0,\zeta_1)}$ into $L^p_{H(\zeta_0,\zeta_1)}$; it is understood that the domain of π contains both $L^p_{\zeta_0}$ and $L^p_{\zeta_1}$. We require also that

$$M \leqq C \max\{M_0, M_1\} \tag{1}$$

for some constant C, where M_0, M_1, M are the corresponding operator norms, $M_0 = \sup\{\|\pi a\|_{L^p_{\zeta_0}}/\|a\|_{L^p_{\zeta_0}}\}$ etc. (It is intended that this should hold for all X, μ, ζ_0, ζ_1, in particular C should only depend on H.) If $C=1$ we say (following DONOGHUE [1]) that H is an *exact interpolation function.* E.g. $z_0^{1-\theta} z_1^\theta$ with $0<\theta<1$ is an exact interpolation function by the well-known theorem of STEIN and WEISS [4]. The first general criterion for a function to be an exact interpolation function was given by FOIAŞ and LIONS [2]. In [3] we gave a somewhat novel deduction of their condition and supplemented it by a new condition in a sense dual to the first one.

For technical reasons mainly we shall restrict below the notion of interpolation function further: We shall require that $H(z_0, z_1)$ is homogeneous of degree 1 and moreover that

$$\lim_{z_0\to 0} H(z_0, z_1) = 0, \qquad \lim_{z_1\to 0} H(z_0, z_1) = 0. \tag{2}$$

We shall give necessary and sufficient conditions for a function to be a (not necessarily exact) interpolation function in the above restricted sense. It will be clear that this settles the problem of interpolation functions for most practical purposes. We note however that within the narrower class of *exact* interpolation function the question is still open, except when $p=2$ [1], [2], but this is now of mostly theoretical interest only.

Let us say that two positive functions u and v defined in any set are *equivalent* if there exists a constant C, $0<C<\infty$, such that $u \leqq Cv$ and $v \leqq Cu$ in that set.

It is clear that if H is an interpolation function then every equivalent function (homogeneous of degree 1) is also an interpolation function. Indeed, this follows from the fact that L^p_ζ is not changed if we replace ζ by an equivalent function.

Theorem 1. *Either of the two following conditions is necessary and sufficient for a function $H=H(z_0, z_1)$ (homogeneous of degree 1) to be an interpolation function of power p:*

i) *H is equivalent to a function of the form*

$$H_1(z_0, z_1) = z_0\left[\varphi\left(\left(\frac{z_1}{z_0}\right)^{-q}\right)\right]^{-\frac{1}{q}} \qquad \left(\frac{1}{p}+\frac{1}{q}=1\right) \tag{3}$$

where φ is positive and concave and $\varphi(\sigma)=o(\max(1,\sigma))$ as $\sigma\to 0$ or ∞;

ii) *H is equivalent to a function of the form*

$$H_2(z_0, z_1) = z_0\left[\psi\left(\left(\frac{z_1}{z_0}\right)^{p}\right)\right]^{\frac{1}{p}} \tag{4}$$

where ψ is positive and concave and $\psi(\sigma)=o(\max(1,\sigma))$ as $\sigma\to 0$ or ∞.

Example. If $H(z_0, z_1)=z_0^{1-\theta}z_1^\theta$ with $0<\theta<1$ we may take $\varphi(t)=\psi(t)=t^{\theta}$ and we obtain thus a weak form of the theorem of Stein and Weiss [4].

Proof (sufficiency). Recall first the result of [3]: A function $H=H(z_0, z_1)$ is an exact interpolation function if it is of the form

$$H(z_0, z_1) = H_3(z_0, z_1) = \left[\int_0^\infty (z_0^p + t^p z_1^p)^{-\frac{q}{p}}\, d\xi(t)\right]^{-\frac{1}{q}} \tag{5}$$

or of the form

$$H(z_0, z_1) = H_4(z_0, z_1) = \left[\int_0^\infty (z_0^{-q} + t^{-q} z_1^{-q})^{-\frac{p}{q}}\, d\eta(t)\right]^{\frac{1}{p}} \tag{6}$$

where $\xi(t)$ and $\eta(t)$ are increasing functions and $\dfrac{1}{p}+\dfrac{1}{q}=1$. (In [3] it was assumed that $\xi(t)$ and $\eta(t)$ were absolutely continuous but this is of course immaterial.) We note that in either case (2) is automatically fulfilled.

Let us first consider the case of ii). It is plain that any H_4 is equivalent to a function of the form

$$z_0\left(\int_0^\infty \min\left\{1, \left(\frac{tz_1}{z_0}\right)^p\right\} d\eta(t)\right)^{\frac{1}{p}},$$

because $(z_0^{-q}+t^{-q}z_1^{-q})^{-\frac{1}{q}}$ and $\min(z_0, tz_1)$ are equivalent, and conversely any such a function is equivalent to a H_4 and thus by [3] it is itself and interpolation function. But since $\min\left\{1, \left(\frac{tz_1}{z_0}\right)^p\right\}$ is concave in $\left(\frac{z_1}{z_0}\right)^p$ this is a H_2 by superposition. Hence, after a change of variable, we have to show that every positive concave function $\psi=\psi(\sigma)$ with $\psi(\sigma)=o(\max(1,\sigma))$ as $\sigma\to 0$ or ∞ can be represented in the form

$$\psi(\sigma)=\int_0^\infty \min\{\sigma,\tau\}\,dw(\tau) \tag{7}$$

where $w(\tau)$ is increasing. Starting with formula (7) we obtain easily by integration by parts

$$\psi(\sigma)=\sigma w(\infty)-\int_0^\sigma w(\tau)\,d\tau$$

or by differentiation (at continuity points of w)

$$\psi'(\sigma)=w(\infty)-w(\sigma).$$

Thus, if we normalize w by $w(\infty)=0$, we have

$$w(\sigma)=-\psi'(\sigma). \tag{8}$$

Conversely, if ψ is given defining w by (8) we easily obtain the desired representation (7). Indeed since $\psi(\sigma)=o(1)$ as $\sigma\to 0$ we have

$$\psi(\sigma)=\int_0^\sigma \psi'(\tau)\,d\tau$$

from which follows by integration by parts

$$\psi(\sigma)=\sigma\psi'(\sigma)+\int_0^\sigma \tau\,dw(\tau).$$

But since $\psi(\sigma)=o(\sigma)$ as $\sigma\to\infty$ we have also $\psi'(\sigma)=o(1)$ as $\sigma\to\infty$. So

$$\psi'(\sigma)=\int_\sigma^\infty dw(\tau)$$

and (7) follows. This settles the case of ii).

It remains the case of i). Now it is immediate that any H_3 is equivalent to a function of the form

$$z_0\left(\int_0^\infty \min\left\{1, \left(\frac{tz_1}{z_0}\right)^{-q}\right\} d\xi(t)\right)^{-\frac{1}{q}}$$

which clearly is a H_1. The converse follows at once by adapting the above argument.

P r o o f (necessity). We base our argument on an idea taken over from DONOGHUE [1] ($p=2$).

Assume that the space X is discrete and contains exactly $(n+1)$ points $x, x_1, \ldots, x_n$ each of these points carrying the mass 1. We take $\zeta_0=1$ and $\zeta_1(x)=z$, $\zeta_1(x_i)=z_i$ $(i=1, \ldots, n)$ where $\left(\frac{1}{z}\right)^q$ is assumed to be in the convex closure of $\left(\frac{1}{z_i}\right)^q$ $(i=1, \ldots, n)$. We define a linear mapping π_1 by setting

$$\begin{cases} (\pi_1 a)(x) = \alpha_1 a(x_1) + \ldots + \alpha_n a(x_n) \\ (\pi_1 a)(x_i) = 0 \qquad (i=1, \ldots, n) \end{cases}$$

where $\alpha_1, \ldots, \alpha_n$ are to be determined. The norm of π_1, as a mapping from $L^p_{\zeta_0}$ into $L^p_{\zeta_0}$, is

$$M_0 = \sup_a \left\{ |\alpha_1 a(x_1) + \ldots + \alpha_n a(x_n)| / (|a(x_1)|^p + \ldots + |a(x_n)|^p)^{\frac{1}{p}} \right\} = (\alpha_1^q + \ldots + \alpha_n^q)^{\frac{1}{q}}$$

or, as a mapping from $L^p_{\zeta_1}$ into $L^p_{\zeta_1}$,

$$M_1 = \sup_a \left\{ z |\alpha_1 a(x_1) + \ldots + \alpha_n a(x_n)| / (|z_1 a(x_1)|^p + \ldots + |z_n a(x_n)|^p)^{\frac{1}{p}} \right\}$$

$$= z \left(\left(\frac{\alpha_1}{z_1}\right)^q + \ldots + \left(\frac{\alpha_n}{z_n}\right)^q \right)^{\frac{1}{q}},$$

this is an immediate consequence of HÖLDER's inequality. We choose now $\alpha_1, \ldots, \alpha_n$ such that $M_0 = M_1 = 1$, i. e.

$$\alpha_1^q + \ldots + \alpha_n^q = 1, \qquad \left(\frac{\alpha_1}{z_1}\right)^q + \ldots + \left(\frac{\alpha_n}{z_n}\right)^q = \left(\frac{1}{z}\right)^q,$$

which is possible since $\left(\frac{1}{z}\right)^q$ is in the convex closure of $\left(\frac{1}{z_i}\right)^q$ $(i=1, \ldots, n)$. But the norm of π_1 as a mapping from $L^p_{H(\zeta_0, \zeta_1)}$ into $L^p_{H(\zeta_0, \zeta_1)}$ is

$$M = H(1, z) \left[\left(\frac{\alpha_1}{H(1, z_1)}\right)^q + \ldots + \left(\frac{\alpha_n}{H(1, z_n)}\right)^q \right]^{\frac{1}{q}}.$$

So if $H(z_0, z_1)$ is an interpolation function we get from (1)

$$\left(\frac{\alpha_1}{H(1, z_1)}\right)^q + \ldots + \left(\frac{\alpha_n}{H(1, z_n)}\right)^q \leqq C \left(\frac{1}{H(1, z)}\right)^q.$$

Thus writing

$$H(1, z) = [\varphi(z^{-q})]^{-\frac{1}{q}}, \quad \sigma = z^{-q}, \quad \sigma_i = z_i^{-q}, \quad \lambda_i = \alpha_i^q \qquad (i=1, \ldots, n)$$

we get

$$(9) \qquad \begin{cases} \lambda_1 + \ldots + \lambda_n = 1, \\ \lambda_1 \sigma_1 + \ldots + \lambda_n \sigma_n = \sigma, \\ \lambda_1 \varphi(\sigma_1) + \ldots + \lambda_n \varphi(\sigma_n) \leqq C \varphi(\sigma) \end{cases}$$

for any $\sigma, \sigma_1, \ldots, \sigma_n$ with σ in the convex closure of $\sigma_1, \ldots, \sigma_n$. It follows now readily that φ is equivalent to a concave function. (If φ satisfies (9) then $\varphi^*(\sigma) = \sup \{\lambda_1 \varphi(\sigma_1) + \ldots + \lambda_n \varphi(\sigma_n)\}$ is concave, i.e. satisfies (9) with $C=1$.) Thus we have shown that every interpolation function is equivalent to a function of the form (3) so i) is a necessary condition.

The necessity of (4) can be proven in a similar fashion by using a linear mapping π_2 defined by a "dual" condition

$$(\pi_2 a)(x) = 0, \qquad (\pi_2 a)(x_i) = \beta_i a(x_i) \qquad (i = 1, \ldots, n)$$

where $\beta_1, \ldots, \beta_n$ are to be determined. We leave the details to the reader.

The proof of theorem 1 is complete.

By making vary p we obtain from theorem 1 the following somewhat surprising byproduct.

Corollary. *Let $\chi = \chi(\sigma)$ be a positive concave function with $\chi(\sigma) = o(\max\{1, \sigma\})$ as $\sigma \to 0$ or ∞. Then for any real number $r \neq 0$ the function $\chi^*(\sigma) = (\chi(\sigma^r))^{1/r}$ is equivalent to a concave function.*

(χ^* may itself not concave (in general), unless $0 < r < 1$.)

Strangely enough we have not been able to give a direct proof of this corollary which does not contain at least some idea from the theory of interpolation spaces.

In the light of the above corollary we can restate theorem 1 in the following more general form:

Theorem 1′. *A function $H = H(z_0, z_1)$ is an interpolation function of power p if and only for some $r \neq 0$ it is equivalent to a function of the form*

$$z_0 \left[\chi\left(\left(\frac{z_1}{z_0} \right)^r \right) \right]^{\frac{1}{r}}$$

where χ is positive and concave and $\chi(\sigma) = o(\max\{1, \sigma\})$ as $\sigma \to 0$ or ∞.

It follows in particular that if a function is an interpolation function of power p for some p then it is so for all p, $1 < p < \infty$.

We conclude by the following

Remark. It is easy to see that if $F(z_0, z_1)$ is an interpolation function of power p then $1/F\left(\frac{1}{z_0}, \frac{1}{z_1}\right)$ is an interpolation function of power q and vice versa. (We owe this observation to J. L. Lions.) In particular we see that (5) and (6) follow directly from each other. Using this circumstance the above proof can be somewhat simplified.

References

[1] W. Donoghue, The interpolation of quadratic norms. *To appear.*

[2] C. Foiaş and J. L. Lions, Sur certains théorèmes d'interpolation, *Acta Sci. Math.*, **22** (1961), 269—282.

[3] J. Peetre, On an interpolation theorem of Foiaş and Lions, *Acta Sci. Math.*, **25** (1964), 255—261.

[4] E. M. Stein and G. Weiss, Interpolation of operators with change of measures, *Trans. Amer. Math. Soc.*, **87** (1958), 159—172.

(Received February 10, 1966)

The algebraic structure of non self-adjoint operators

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The purpose of this paper is to give an algebraic approach to the theory of non self-adjoint operators on (complex) Hilbert space by means of the theory of von Neumann algebras. In the spectral theory, the principal problem is to reduce a given operator to simpler operators. We shall consider this problem, from the algebraic view point, for a certain class of non self-adjoint operators.

Let A be an operator on Hilbert space. We shall denote by $R(A)$ the von Neumann algebra generated by A (i.e., the smallest von Neumann algebra containing A) and we say that A is *primary* if $R(A)$ is a factor. Then the spectral decomposition of a normal operator A essentially means the decomposition of A into primary normal operators (which are scalar operators). Moreover, we know that an isometry is decomposed into the direct sum of a unitary operator and a unilateral shift. As we have shown in [9; Lemma 2] (cf. [3; Theorem 1]), a unilateral shift is a primary operator. From this fact, we can easily see that a non-scalar isometry is a unilateral shift if and only if it is primary. Therefore, the decomposition of an isometry V mentioned above is essentially that of V into primary isometric operators whith the aid of the spectral theorem for a unitary operator. From this point of view, the decomposition of an operator A into primary operators may be regarded as a kind of spectral decomposition of A.

We shall concern ourselves with the class of operators whose imaginary parts are completely continuous. M. S. BRODSKIĬ and M. S. LIVŠIC, cf. [1], [5], have developed a theory of the triangular form for operators whose imaginary parts belong to the trace class. Our purpose is to establish the decomposition of an operator with completely continuous imaginary part into primary operators belonging to the same class and to show that a primary operator of this class is the direct sum of copies of an irreducible operator of the same class by making use of the theory of von Neumann algebras. Consequently, we shall be able to see some algebraic aspects of operators with completely continuous imaginary part. This paper contains the details of the research announcement appeared in [8].

For the sake of simplicity, we shall assume that our Hilbert space is separable. By an operator we always understand a bounded linear transformation on a Hilbert space. By a von Neumann algebra we understand a self-adjoint operator algebra with the identity operator I which is closed in the weak topology. The set of operators each of which commutes with every operator in a von Neumann algebra M will

*) This research was partially supported by the National Science Foundation (NSF Contract G 24295) at the University of Minnesota (1964—1965).

be called the commutant of M and be denoted by M'. The commutant M' is again a von Neumann algebra and $M=M''$. A factor means a von Neumann algebra whose center consists of scalar multiples of the identity operator. For terminology, notation and basic results, we shall refer to the book of J. DIXMIER [2].

1. The structure of operators with completely continuous imaginary part

In this section, we shall restrict our consideration to an operator A on a Hilbert space H whose imaginary part $\mathrm{Im}\,(A)=\frac{1}{2i}(A-A^*)$ is completely continuous. Our object is to prove the following

Theorem 1. *An operator A with completely continuous imaginary part on a Hilbert space H is decomposed by a unique countable family of mutually orthogonal central projections P_0, P_i $(i\in I)$ in $R(A)$ into the form*

$$A = A_{P_0}\oplus \sum_{i\in I}\oplus A_{P_i},$$

where the restriction A_{P_0} of A to P_0H is a self-adjoint operator, the restriction A_{P_i} of A to P_iH $(i\in I)$ is a primary operator with completely continuous imaginary part and $P=\sum_{i\in I} P_i$ is the projection on the subspace generated by vectors of the form $A^n\varphi$ $(\varphi\in \mathrm{Im}\,(A)H;\ n=0, 1, 2, \ldots)$.

Certainly the essence of our result is in the reduction theory of VON NEUMANN [6], that is, in the direct integral decomposition of $R(A)$ into factors, but it should be noticed that the character of the operator A has induced a more simple and concrete decomposition of $R(A)$. Before beginning the proof, we shall provide some lemmas. We shall denote by K the range of $\mathrm{Im}\,(A)$, i. e.,

$$K = \frac{1}{2i}(A-A^*)H$$

and the projection on the subspace $\bar{K}$ will be denoted by E. In what follows, M always means the von Neumann algebra $R(A)$ generated by A. Since $\mathrm{Im}\,(A)$ is a self-adjoint completely continuous operator, it is well known that there exists an orthonormal basis in H whose elements are proper vectors of $\mathrm{Im}\,(A)$. Therefore, if we denote by $\{\mu_k\}$ $(k\in N)$ the countable family of all distinct non-zero proper values of $\mathrm{Im}\,(A)$ and by E_k the projection on the proper subspace corresponding to μ_k, each E_kH is finite dimensional and $E=\sum_{k\in N} E_k$. As the first step, we observe that each projection E_k belongs to M and hence the projection E is in M. This notable fact is the direct consequence of the following

Proposition 1. *Let B be an operator in the von Neumann algebra M. Then a projection on a proper subspace of B belongs to M.*

Proof. Let μ be a proper value of B and let $\mathcal{N}(\mu)$ the proper subspace corresponding to μ. We denote by F the projection on $\mathcal{N}(\mu)$. In order to prove that F

belongs to M, it is sufficient to show that F commutes with all operators belonging to the commutant M' of M. Let A' be an arbitrary operator in M'. Then, for every vector φ in the proper subspace $\mathcal{N}(\mu)$, the equality $B(A'\varphi)=A'B\varphi = A'\mu\varphi=\mu A'\varphi$ yields $A'\varphi\in\mathcal{N}(\mu)$. Similarly we have $A'^{*}\varphi\in\mathcal{N}(\mu)$ for every vector $\varphi\in\mathcal{N}(\mu)$. Thus the subspace $\mathcal{N}(\mu)$ reduces A'. This means that F commutes with A'.

We consider the subspace H_1 generated by vectors of the form $A^n\varphi$ ($\varphi\in K$; $n=0, 1, 2, \ldots$) and denote by P the projection on H_1. As is well known, the projection P plays a very important role in the study of our operator A, and so we need to find the exact relation between A and P.

Lemma 1. *The subspace H_1 coincides with the subspace $[MK]$ generated by vectors of the form $B\varphi$ ($B\in M$, $\varphi\in K$). That is, the projection P belongs to the center of M.*

Proof. It is clear that H_1 is invariant by A, and so its orthogonal complement $H_2=H\ominus H_1$ is invariant by A^*. For each vector $\varphi\in H_2$, we have $\left\langle \frac{1}{i}(A-A^*)\varphi, \psi\right\rangle = \left\langle \varphi, \frac{1}{i}(A-A^*)\psi\right\rangle=0$ for every vector $\psi\in H$. Thus $A\varphi=A^*\varphi$ for every vector $\varphi\in H_2$. This means that the subspace H_2 is invariant by A. Therefore, the subspace H_2 reduces A, that is to say, the projection Q on H_2 belongs to M'. Thus the projection $P=I-Q$ belongs to M'. For each operator $B\in M$ and for each vector $\varphi\in K$, the equality $B\varphi=BP\varphi=PB\varphi$ implies $[MK]\subset H_1$. On the other hand, obviously H_1 is contained in $[MK]$. Consequently, we obtain that the subspace H_1 coincides with the subspace $[MK]$.

Next we observe that the subspace $[MK]$ reduces every operator $B'\in M'$. In fact, since the projection E is in M by Proposition 1, $B'MK=MB'K=MB'EK=MEB'K\subset MK$ for each operator $B'\in M'$. In the same way, we can get $B'^{*}MK\subset MK$. It follows from this fact that P commutes with every operator belonging to M'. Thus the projection P belongs to M. Consequently, the projection P belongs to the center $M\cap M'$ of M.

Remark. In the theory of von Neumann algebras, the projection on $[MK]$ is called the central support of E. Indeed, it is the minimal central projection containing E. We have shown that the projection P is the central support of the projection E.

The following lemma on von Neumann algebras is essentially known, but, for the sake of completeness, we shall give the proof.

Lemma 2. *Let F be a minimal projection in M [1]) with the central support R. Then there exists a countable family of orthogonal, equivalent projections $\{F_j\}$ ($j\in J$) such that $R=\sum_{j\in J}F_j$ and $F_{j_0}=F$ for a fixed $j_0\in J$.*

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[1]) A minimal projection in M means a non-zero projection F belonging to M such that $G\leqq F$ and $O\neq G\in M$ implies $G=F$.

Proof. Let $F_j\,(j\in J)$ be a miximal family of orthogonal, equivalent projections such that $F_{j_0}=F$. Then $F_j\leqq R$ for all $j\in J$. Put $G=R-\sum_{j\in J}F_j$. By using the theorem of comparison (cf. [1: Ch. III, Theorem 1]), we can find a central projection Q such that

$$GQ\prec FQ \quad\text{and}\quad F(I-Q)\prec G(I-Q).$$

If $F(I-Q)\neq O$, $F(I-Q)=F$ since F is minimal in M. It follows that $F\leqq I-Q$ and $F\prec G(I-Q)\leqq G$. This contradicts to the maximality of $\{F_j\}$ $(j\in J)$. Thus $F(I-Q)$ must be zero, and so $F\leqq Q$. Then we have $GQ\prec F$. Since F is minimal in M, $GQ=O$ or $GQ\sim F$. Therefore, $GQ=O$ since $GQ\sim F$ obviously yields the contradiction. It follows that

$$O=GQ=RQ-\sum_{j\in J}F_jQ=RQ-\sum_{j\in J}F_j.$$

Keeping in mind that R is the central support of F, we get

$$R=RQ=\sum_{j=J}F_j.$$

Proof of Theorem 1. From Lemma 1 it follows that the operator A is decomposed by the central projection P into the form

$$A=A_{I-P}\oplus A_P$$

where A_{I-P} is a self-adjoint operator on $(I-P)H$ and P is the central support of E. As we have already seen, the projection E is expressed as the direct sum of finite dimensional projections E_k $(k\in N)$ in M. Since each projection E_k is finite dimensional and P is the central support of E, we can choose a family of minimal projections F_i $(i\in I)$ in M contained in some of E_k such that the central supports P_i of F_i are mutually orthogonal and $P=\sum_{i\in I}P_i$. By making use of Lemma 2, we can get a family of orthogonal, equivalent projections F_{ij} $(j\in J)$ such that $F_i\doteq F_{ij_0}$ and $P_i=\sum_{j\in J}F_{ij}$. Then the restriction M_{P_i} of M to P_iH is spatially isomorphic to $M_{F_i}\otimes\mathscr{L}(l_2(J))$[2]) where M_{F_i} is the restriction of M to F_iH and $\mathscr{L}(l_2(J))$ means the algebra of all operators on $l_2(J)$. Since F_i is minimal in M, M_{F_i} is the scalar multiples of the identity operator on F_iH and hence M_{P_i} is a factor. Note that P_i is a central projection. Then we obtain that the factor M_{P_i} is generated by A_{P_i}, that is to say, each operator M_{P_i} is a primary operator. In addition, it is obvious that $\operatorname{Im}(A_{P_i})$ is completely continuous. Putting $P_0=I-P$, we obtain the desired result since the uniqueness of a family $\{P_0, P_i\}$ $(i\in I)$ directly follows from the fact that each operator A_{P_i} $(i\in I)$ is primary.

What our theorem means is quite well illustrated by taking a normal operator of this class. Indeed, Theorem 1 yields the spectral decomposition of the non self-adjoint part of this operator.

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[2]) The notation $\otimes$ always means the tensor product of Hilbert spaces, operators, or von Neumann algebras.

Corollary 1. *Let A be a normal operator with completely continuous imaginary part. Then A is uniquely expressed by a countable family of mutually orthogonal projections P_0, P_i $(i\in I)$ in $R(A)$ as follows:*

$$A = AP_0 + \sum_{i\in I} \lambda_i P_i,$$

where each P_i $(i\in I)$ is finite dimensional and $I = P_0 + \sum_{i\in I} P_i$, moreover $\{\lambda_i\}$ $(i\in I)$ is a family of non-real proper values of A and AP_0 is a self-adjoint operator.

In fact, since A is normal, each operator A_{P_i} in Theorem 1 must be a scalar operator $\lambda_i I_i$ (where I_i is the identity operator on P_iH). Furthermore, since A_{P_i} is a non self-adjoint operator (in this case $P_i = F_i \leqq E_k \leqq E$) and has a completely continuous imaginary part, each λ_i is a non-real number and P_iH must be finite dimensional. Then, by Theorem 1, $A = AP_0 + \sum_{i\in I} \lambda_i P_i$ and clearly λ_i is a proper value of A. Thus our result is the decomposition of the non-real spectrum of this operator.

Next we shall mention a very important special class of our operators. That is, we shall consider the class of operators whose imaginary parts are finite dimensional operators. Let A be an operator with finite dimensional imaginary part. Then the dimension r of the range of $\mathrm{Im}(A)$ is called the *non-hermitian rank* of A. In this case, Theorem 1 may be stated as follows.

Corollary 2. *An operator A with non-hermitian rank r is decomposed by a unique family of mutually orthogonal central projections $P_0, P_1, \ldots, P_n$ $(n \leqq r)$ in $R(A)$ into the form*

$$A = A_{P_0} \oplus A_{P_1} \oplus \ldots \oplus A_{P_n},$$

where A_{P_0} is a self-adjoint operator and $\{A_{P_1}, \ldots, A_{P_n}\}$ is a family of primary operators with non-hermitian rank k_i such that

$$\sum_{i=1}^{n} k_i = r.$$

In fact, from the proof of Theorem 1 we can easily see that a family P_i $(i\in I)$ is finite and the non-hermitian rank k_i of A_{P_i} $(i = 1, 2, \ldots, n)$ is equal to $\dim(EP_i)$. Accordingly we have

$$\sum_{i=1}^{n} k_i = \sum_{i=1}^{n} \dim(EP_i) = \dim\left(E \sum_{i=1}^{n} P_i\right) = \dim(EP) = \dim(E) = r.$$

2. The algebraic type of operators with completely continuous imaginary part

The structure of an operator A is closely related to the type of the von Neumann algebra $R(A)$ generated by A. An operator A is said to be of type I if $R(A)$ is of type I and moreover a primary operator A is said to be of type I_n (resp. type I_∞) if the factor $R(A)$ is of type I_n (resp. type I_∞). Then the question coming to our mind is this: which non-normal operators are of type I? Partial answers to this question are known. We know that an isometry is of type I ([8]). Moreover, we have shown that a completely continuous operator is of type I ([8]). This result will be generalized in what follows. Indeed, from the proof of Theorem 1 it is easy to determine the type of operators with completely continuous imaginary part.

Theorem 2. *An operator A with completely continuous imaginary part is of type I.*

Proof. As we have seen in the proof of Theorem 1, each operator A_{P_i} generates a von Neumann algebra M_{P_i} of type I_α ($\alpha = n$ or ∞) (recall that M_{P_i} is spatially isomorphic to $(\lambda I_i) \otimes \mathscr{L}(l_2(J))$). Moreover, since $\{P_0, P_i\}$ $(i \in I)$ is a family of mutually orthogonal central projections, the von Neumann algebra $M = R(A)$ is decomposed as the direct sum

$$M = M_{P_0} \oplus \sum_{i \in I} \oplus M_{P_i}.$$

Thus we can conclude the theorem since the abelian von Neumann algebra M_{P_0} is of type I (cf. [1; Ch. I, § 8, Prop. 1]).

Here is a very remarkable fact which illustrates the algebraic aspect of primary operators of our class. In our decomposition, it is possible that A_{P_i} has the type I_∞ (actually we may restrict our attention to this case), but the commutant $R(A_{P_i})'$ of the von Neumann algebra $R(A_{P_i})$ has necessarily the type I_n ($n = 1, 2, \ldots$). To show this it is sufficient to consider only a non-scalar primary operator A. Let A be a non-scalar primary operator with completely continuous imaginary part. Then $R(A)$ contains obviously a finite dimensional minimal projection (recall that each projection on a proper subspace of Im (A) corresponding to a non-zero proper value is finite dimensional). Since all minimal projections in the factor $R(A)$ are equivalent to each other, the dimension d of a minimal projection in $R(A)$ is uniquely determined by the operator A. In what follows, the dimension d will be called the *multiplicity* of the operator A.

Proposition 2. *Let A be a non-scalar primary operator with completely continuous imaginary part. Then the commutant $R(A)'$ of $R(A)$ is of type I_n where n is the multiplicity of A.*

Proof. Let F be a minimal projection in $M = R(A)$. Then $\dim(F) = n$. By Lemma 2, we can choose a family of mutually orthogonal, equivalent projections $\{F_j\}$ $(j \in J)$ in M such that $F_{j_0} = F$ and $\sum_{j \in J} F_j = I$. Then M is spatially isomorphic to $M_F \otimes \mathscr{L}(l_2(J))$. The minimality of F implies that M_F is the scalar multiples of the identity operator on FH. Thus M' is spatially isomorphic to $(M_F)' \otimes \mathscr{L}(l_2(J))' = \mathscr{L}(FH) \otimes \mathscr{C}$ where $\mathscr{C}$ is the von Neumann algebra of scalar multiples of the identity operator on $l_2(J)$. Since $\mathscr{L}(FH)$ is of type I_n, $\mathscr{L}(FH) \otimes \mathscr{C}$ is of type I_n. Therefore, M' is of type I_n.

Corollary. *A primary operator A with non-hermitian rank* 1 *is irreducible (i.e., A has no non-trivial reducing subspace).*

3. The decomposition of a primary operator into irreducible operators

In this section, we shall show that a primary operator A with completely continuous imaginary part is expressed as the direct sum of copies of an irreducible operator of the same class. Indeed, Proposition 2 makes it possible to decompose A into irreducible operators in a simple way. Consequently, the study of our operators may be reduced to the case of irreducible operators of our class. We shall mention here some examples of irreducible operators with completely continuous imaginary part.

Example 1. The simplest irreducible operator with non-hermitian rank 1 on an infinite dimensional Hilbert space is the integral operator on $L_2(0, 1)$ defined by

$$(Af)(x) = i\int_0^x f(t)\,dt.$$

Indeed, $L_2(0, 1)$ is generated by the vectors of the form $A^n\varphi$ $(n=0, 1, 2, \dots)$ where $\varphi(x)\equiv 1$, and the range of $\operatorname{Im}(A)$ consists of scalar multiples of the vector φ. Thus it follows from Theorem 1 that A is primary. By what was already seen in the preceding section the operator A is irreducible. Moreover, we know that this operator is quasi-nilpotent. Here we should mention that the integral operator A is characterized by these algebraic properties. That is, the notable result obtained in [2] and [4] may be stated as follows: a quasi-nilpotent primary operator with non-hermitian rank 1 is unitarily equivalent (up to a non-zero real scalar multiple) to the integral operator A on $L_2(0, 1)$.

Example 2. Let V be a unilateral shift on a Hilbert space H. That is, for an orthonormal basis $\{\varphi_n\}$ $(n=1, 2, \dots)$ in H, $V\varphi_n=\varphi_{n+1}$ for all n. Now we consider an operator A of the form VB, where B is a positive completely continuous operator whose range spans H. We shall show that the operator A is irreducible. Put $M=R(A)$. Then the equality $B^2=BV^*VB=(VB)^*(VB)=A^*A$ implies $B^2\in M$. Thus $B=(B^2)^{1/2}$ belongs to M. Here, B is expressed in the form: $B=\sum_n \lambda_n E_n$, where $\{\lambda_n\}$ is the countable family of all distinct proper values of B, E_n is the projection on the proper subspace corresponding to λ_n and $\sum_n E_n=I$. Since the range of B spans H, $\lambda_n>0$ for all n. By Proposition 1, each projection E_n belongs to M. Hence, for each k we have

$$\lambda_k VE_k = V\Big(\sum_n \lambda_n E_n\Big)E_k = VBE_k\in M,$$

and so $VE_k\in M$ for each k. Consequently, $V=V\big(\sum_n E_n\big)=\sum_n VE_n\in M$. Since V is irreducible, $R(V)=\mathscr{L}(H)$ is contained in M. Thus $M=\mathscr{L}(H)$, that is to say, A is irreducible.

Theorem 3. *Let A be a non-scalar primary operator with completely continuous imaginary part and let m be the multiplicity of A. Then A is unitarily equivalent to an operator $V\otimes I_m$, where V is an irreducible operator with completely continuous imaginary part and I_m is the identity operator on an m-dimensional Hilbert space. In particular, if A has the non-hermitian rank r, A is unitarily equivalent to $V\otimes I_m$, where V is an irreducible operator with non-hermitian rank n and $r=mn$.*

Proof. We shall take here the projection E, P in $M=R(A)$ as in the section 1. From Proposition 2 it follows that there exists a family of mutually orthogonal, equivalent (minimal) projections $P_1, P_2, \ldots, P_m$ in M' such that $I=\sum_{i=1}^{m} P_i$. Then each operator A_{P_i} ($i=1, 2, \ldots, m$) is an irreducible operator with completely continuous imaginary part. In fact, since P_i belongs to M', $A_{P_i}-A^*_{P_i}=(A-A^*)_{P_i}$. To see that A_{P_i} is irreducible we consider the von Neumann algebra M_{P_i} which is clearly generated by A_{P_i}. As is well known, $(M_{P_i})'=M'_{P_i}$. Here the right-hand side consists of scalar multiples of the identity operator on P_iH since P_i is a minimal projection in M'. This means that A_{P_i} is irreducible.

Let W_i be a partially isometric operator in M' such that $W_i^*W_i=P_i$ and $W_iW_i^*=P_1$. Then, for every vector $\varphi \in P_1H$,

$$W_iA_{P_i}W_i^*\varphi = W_iAP_iW_i^*\varphi = W_iAW_i^*\varphi = AW_iW_i^*\varphi = AP_1\varphi = A_{P_1}\varphi.$$

Thus each operator A_{P_i} is unitarily equivalent to A_{P_1} by W_i. Now the assertion will be completed by the standard argument. Put $\mathfrak{H}=P_1H$ and $V=A_{P_1}$, then, as is well known, $H=\mathfrak{H}\otimes l_2(N)$ where $N=\{1, 2, \ldots, m\}$, and each vector $\varphi\in\mathfrak{H}\otimes l_2(N)$ is expressed in the form:

$$\varphi = \sum_{i=1}^{m} \varphi_i \otimes \varepsilon_i,$$

where $\{\varepsilon_i\}$ ($i=1, 2, \ldots, m$) is an orthonormal basis of $l_2(N)$ and $\{\varphi_i\}$ ($i=1, 2, \ldots, m$) is a family of vectors in $\mathfrak{H}$. Define a linear transformation W of H onto $\mathfrak{H}\otimes l_2(N)$ as follows:

$$W\varphi = \sum_{i=1}^{m} W_iP_i\varphi_i\otimes\varepsilon_i \quad \text{for each vector} \quad \varphi\in H.$$

Then it is verified by straightforward computation that W is an isometry of H onto $\mathfrak{H}\otimes l_2(N)$ and $W^{-1}\left(\sum_{i=1}^{m}\varphi_i\otimes\varepsilon_i\right) = \sum_{i=1}^{m} W_i^*\varphi_i$. Therefore, we have

$$WAW^{-1}\left(\sum_{i=1}^{m}\varphi_i\otimes\varepsilon_i\right) = W\left(\sum_{i=1}^{m}AW_i^*\varphi_i\right) =$$

$$= \sum_{i=1}^{m} W_iP_i\left(\sum_{j=1}^{m}AW_j^*\varphi_j\right)\otimes\varepsilon_i = \sum_{i=1}^{m} W_iAP_iW_i^*\varphi_i\otimes\varepsilon_i =$$

$$= \sum_{i=1}^{m} A_{P_1}\varphi_i\otimes\varepsilon_i = (V\otimes I_m)\left(\sum_{i=1}^{m}\varphi_i\otimes\varepsilon_i\right).$$

That is, we have shown that A is unitarily equivalent to $V\otimes I_m$.

If A has the non-hermitian rank r, $r=\dim(E)=\sum_{i=1}^{m}\dim(P_iE)$. Since P_iE are equivalent to each other in $\mathscr{L}(H)$, $\dim(P_1E)=\ldots=\dim(P_mE)$. Thus $\dim(P_1E)= =r/m=n$. Note that $(V-V^*)P_1H=(A_{P_1}-A^*_{P_1})P_1H=P_1(A-A^*)H=P_1EH$. Then we obtain that V has the non-hermitian rank n.

Remark. We cannot express an arbitrary primary operator (not necessarily of type I) as the direct sum of copies of an irreducible operator. That is, from the fact that an operator with completely continuous imaginary part is of type I, our theorem has been effected.

Corollary. *Let A be a primary operator with non-hermitian rank r. Then the multiplicity of A is equal to the rank r if and only if A is unitarily equivalent to $V\otimes I_r$, where V is an irreducible operator with non-hermitian rank 1 and I_r is the identity operator on an r-dimensional Hilbert space.*

Here we shall concentrate our attention on the case when the non-hermitian rank r of a primary operator A is a prime number. Then the multiplicity m of A must be either 1 or r. In the case of $m=1$, A is irreducible. The case of $m=r$ is that of the above corollary. Thus we have the following

Theorem 4. *Let A be a primary operator with non-hermitian rank r. If r is a prime number, then A is either irreducible or unitarily equivalent to $V\otimes I_r$, where V is an irreducible operator with non-hermitian rank 1 and I_r is the identity operator on an r-dimensional Hilbert space.*

In closing this section, to see how our results illustrate the algebraic structure of an operator with finite non-hermitian rank, we shall mention here the special cases of our operators. Actually the structure of an operator A with non-hermitian rank r depends on the dimensions of minimal projections with respect to $R(A)$ contained in the projection E (E: the projection on the range of $\mathrm{Im}(A)$). We shall state the possible forms of operators in the cases of non-hermitian rank $r=1, 2, 3$.

An operator A with non-hermitian rank 1 has the following structure:

$$A=A_s\oplus A_i$$

where A_s is a self-adjoint operator and A_i is an irreducible operator with non-hermitian rank 1.

An operator A with non-hermitian rank 2 has one of the following structures:

(1) $A=A_s\oplus A_i$, (2) $A=A_s\oplus A_{i_1}\oplus A_{i_2}$, (3) $A=A_s\oplus A_j$,

where A_s is a self-adjoint operator; A_i is an irreducible operator with non-hermitian rank 2; A_{i_1} and A_{i_2} are irreducible operators with non-hermitian rank 1; A_j is unitarily equivalent to $V\otimes I_2$ (V, I_2: operators in Theorem 4). (1) and (2) arise in the case when the projection E contains a minimal projection of dimension 1, and (3) arises in the case when the projection E is a minimal projection.

An operator A with non-hermitian rank 3 has one of the following structures:

(1) $A=A_s\oplus A_i$, (2) $A=A_s\oplus A_{i_1}\oplus A_{i_2}$,

(3) $A=A_s\oplus A_{j_1}\oplus A_{j_2}\oplus A_{j_3}$, (4) $A=A_s\oplus A_k$,

where A_s is a self-adjoint operator; A_i is an irreducible operator with non-hermitian rank 3; A_{i_1}, A_{j_1}, A_{j_2}, and A_{j_3} are irreducible operators with non-hermitian rank 1; A_{i_2} is a primary operator with non-hermitian rank 2 (cf. Theorem 4); A_k is unitarily equivalent to $V\otimes I_3$ in Theorem 4. (1), (2) and (3) arise in the case when the projection E contains a minimal projection of dimension 1. (4) arises in the case when the projection E is a minimal projection.

4. Spectral properties

The basic properties of the spectrum of an operator A with completely continuous imaginary part are known. From M. S. BRODSKIĬ—M. S. LIVŠIC [2], we know that every non-real point of the spectrum of the operator A is a proper value and its proper subspace is finite dimensional. Moreover, we know that the set of non-real points of the spectrum of the operator A is at most countable and a limit point of this set is on the real line. If we denote by $\sigma(A)$ (resp. $\sigma_P(A)$) the spectrum (resp. the point spectrum) of A, Theorem 1 yields that $\sigma(A)$ and $\sigma_P(A)$ are divided as follows:

$$\sigma(A)=\sigma(A_0)\cup\big(\bigcup_{i\in I}\sigma(A_i)\big) \quad \text{in case } I \text{ is finite and} \quad \sigma_P(A_0)=\sigma_P(A)\cup\big(\bigcup_{i\in I}\sigma_P(A_i)\big).$$

Hence, in studying the spectrum of our operator, we may concentrate our attention on that of our primary operator. In this case we should point out from Theorem 3 that the (resp. point) spectrum of a primary operator of this class coincides with the (resp. point) spectrum of an irreducible operator of the same class. Here is a significant and interesting problem: how does the algebraic simplicity of a primary operator effect its spectrum? Although many questions about it are left to be settled in the future, we shall have some comments on this subject. The following lemma may be viewed as a step toward our desire.

Lemma 3. *Let A be a non-scalar primary operator. Then every proper value of A lies in the open disc $D=\{\lambda: |\lambda|<\|A\|\}$.*

Proof. Suppose that A has a proper value λ with $|\lambda|=\|A\|$. Then there exists a non-zero vector φ such that $A\varphi=\lambda\varphi$. Put $B=\frac{1}{\lambda}A$. Then $\|B\|=1$ and $B\varphi=\varphi$. From this fact it follows that $B^*\varphi=\varphi$ (cf. [7: Chap. X, No 143]). Consequently, $A^*\varphi=\lambda\varphi$. Thus the proper subspace $\mathcal{M}$ corresponding to λ reduces A. That is, the projection P on $\mathcal{M}$ commutes with A. This means that P belongs to the commutant $R(A)'$ of $R(A)$. As we have already seen in Proposition 1, P is the projection in $R(A)$. Hence P belongs to the center $R(A)\cap R(A)'$. Since the non-zero projection P is not the identity operator by the assumption, this contradicts the fact that A is a primary operator.

Combining the known result mentioned above and Lemma 3, we can conclude the following

Proposition 3. *Every non-real point of the spectrum of a non-scalar primary operator A with completely continuous imaginary part lies in the open disc $D=\{\lambda: |\lambda|<\|A\|\}$.*

Now let us consider an operator A with non-hermitian rank 1 whose spectrum is real. We shall show that our decomposition (Theorem 1) induces the spectral decomposition of A in the sense that A is decomposed by a central projection in $R(A)$ into the form $A=B\oplus C$ where B (resp. C) has a pure point (resp. continuous) spectrum.[3])

[3]) If $\sigma(A)$ coincides with the point (resp. continuous) spectrum of A, we say that A has a pure point (resp. continuous) spectrum.

Lemma 4. *Let A be a primary operator with non-hermitian rank* 1 *and let λ be a real scalar. Then the range of $A-\lambda I$ is dense in H.*

Proof. Let $\mathscr{M}$ be the range of $A-\lambda I$ and let $\mathscr{N}$ be the orthogonal complement of $\mathscr{M}$. Then, for each pair of vectors φ, ψ in $\mathscr{N}$, we have

$$\langle (A-A^*)\varphi, \psi\rangle = \langle [(A-\lambda I)-(A^*-\lambda I)]\varphi, \psi\rangle =$$

$$= \langle (A-\lambda I)\varphi, \psi\rangle - \langle \varphi, (A-\lambda I)\psi\rangle = 0.$$

Since $K=\mathrm{Im}\,(A)H$ is one dimensional, the subspace $[(A-A^*)\mathscr{N}]$ must be K or $\{0\}$. Therefore, $\mathscr{N}$ is contained in $H\ominus K$. Keeping in mind that $A\psi=A^*\psi$ for every vector $\psi\in H\ominus K$, we have $A\varphi=A^*\varphi$ for every vector $\varphi\in\mathscr{N}$. This implies that $\mathscr{N}$ reduces A since $\mathscr{N}$ is invariant by $A^*-\lambda I$, i.e., A^*. In other words, the projection P on $\mathscr{N}$ belongs to $R(A)'$. As we have seen in the section 2, A is irreducible. Thus P must be I or O. But obviously $P\neq I$, and so $P=O$, that is, $\mathscr{N}=\{0\}$. This states that $\mathscr{M}$ is dense in H.

Lemma 5. *A primary operator A with non-hermitian rank* 1 *does not have a real proper value.*

Proof. Suppose that A has a real proper value λ. Let $\mathscr{M}$ be a proper subspace of A corresponding to λ. Then it is immediately seen that $\mathscr{M}$ is orthogonal to the range of $A^*-\lambda I$. Since A^* has also the non-hermitian rank 1, the range of $A^*-\lambda I$ is dense in H by Lemma 4. Thus $\mathscr{M}$ contains only the zero vector, which is contradiction.

Proposition 4. *Let A be an operator with non-hermitian rank* 1 *whose spectrum is real. Then A is decomposed by a central projection R in $R(A)$ into the form:*

$$A = A_R \oplus A_{I-R},$$

where A_R has a pure point spectrum and A_{I-R} has a pure continuous spectrum.

Proof. By Theorem 1, Corollary 2, A is decomposed by a central projection P in $R(A)$ into the form:

$$A = A_P \oplus A_{I-P},$$

where A_P is a self-adjoint operator and A_{I-P} is a primary operator with non-hermitian rank 1. Furthermore, as is well known, the self-adjoint operator A_P is decomposed by a projection R in $R(A_P)$ (which is a central projection in $R(A)$) as follows:

$$A_P = A_R \oplus A_{P-R},$$

where A_P has a pure point spectrum and A_{P-R} has a pure continuous spectrum. Since the spectrum of A_{I-P} is real, the preceding lemmas mean that A_{I-P} has a pure continuous spectrum. Consequently, we can easily see that $A_{I-R} = A_{P-R}\oplus A_{I-P}$ has a pure continuous spectrum.

References

[1] M. S. Brodskiĭ and M. S. Livšic, Spectral analysis of non self-adjoint operators and intermediate systems, *Uspehi Mat. Nauk,* **13** (1958), 3—85 (Russian); *Amer. Math. Soc. Transl.,* (2) **13** (1960), 265—346.
[2] J. Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien* (Paris, 1957).
[3] P. R. Halmos, Shifts on Hilbert spaces, *J. reine angew. Math.,* **208** (1961), 102—112.
[4] G. K. Kalisch, Direct proofs of spectral representation theorems, *J. Math. Anal. and Appl.,* **8** (1964), 356—363.
[5] M. S. Livšic, On the spectral decomposition of linear non self-adjoint operators, *Mat. Sbornik,* **34** (1956), 145—198 (Russian); *Amer. Math. Soc. Transl.,* (2) **5** (1957), 67—114.
[6] J. von Neumann, On rings of operators. Reduction theory, *Ann. of Math.,* (2) **50** (1949), 401—485.
[7] F. Riesz and B. Sz.-Nagy, *Leçons d'analyse fonctionnelle* (Budapest, 1952).
[8] N. Suzuki, Algebraic aspects of non self-adjoint operators, *Proc. Jap. Acad.,* **41** (1965), 706—710.
[9] N. Suzuki, Isometries on Hilbert spaces, *Proc. Jap. Acad.,* **39** (1963), 435—438.
[10] N. Suzuki, On the type of completely continuous operators, *Proc. Jap. Acad.,* **40** (1964), 683—685.

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(Received February 9, 1966)

Homomorphisms of a semigroup onto normal bands

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1. Introduction and summary

Homomorphisms of an arbitrary semigroup S onto semigroups belonging to a special class $\mathscr{C}$ of semigroups furnish some information about the structure of S. Of particular interest is the case when $\mathscr{C}$ is a class of bands, for in such a case all the classes of the decomposition of S induced by such homomorphisms are subsemigroups of S. The problem of finding a sufficiently explicit characterization of homomorphisms onto arbitrary bands appears difficult. In [3] and [4], we have found such a characterization for the case when $\mathscr{C}$ is the class of all semilattices and the class of all rectangular bands, respectively. In the present work we solve the problem when $\mathscr{C}$ is the class of all [left] normal bands (for definitions see section 2). Normal bands have been studied by YAMADA and KIMURA [9], and right normal bands by VAGNER [8], and SHAIN [5], [6] (the latter two call right normal bands "restrictive semigroups").

In section 2 we give the definitions and notation used in the paper. In it we introduce a number of concepts which are used in succeeding sections; section 3 discusses some of their properties. Section 4 contains main results of the paper, viz., characterization of congruences induced by homomorphisms onto [left] normal bands. In section 5 we discuss some general properties of normal bands and, in section 6, subdirect products of such bands. Section 7 contains a representation of normal bands as subsets of a set under certain multiplication.

Some of the results in this paper parallel those in [3] and [4] (similar methods are used); we will not expressly mention similarity with these papers.

2. Definitions and notation

Throughout S will denote an arbitrary semigroup unless stated otherwise. A one element set $X=\{x\}$ will be simply denoted by x. For properties of most of the concepts that are introduced below see [3] and [4]. Let H be a non-empty subset of S and let $x, y, z \in S$.

H is said to be *left (right) dense* if $xy \in H$ implies $x \in H$ ($y \in H$), *quasi dense* if (i) $x^2 \in H$ implies $x \in H$ and (ii) $xz \in H$ if and only if $xyz \in H$. A left dense and right dense subsemigroup of S is called a *face* of S. The smallest face of S containing H is denoted by $N(H)$. H is said to be a *left (right) normal complex* (abbreviated l. n.

complex [r. n. complex]) if H is a left dense right ideal (l.d.r.i.) [right dense left ideal (r.d.l.i.)] of $N(H)$. H ist said to be a *normal complex* (n complex) if H is a quasi dense subsemigroup of $N(H)$. By $A(x)$ [$B(x)$] denote the smallest l.n. [r.n.] complex of S containing x. (It is easy to see that the non-empty intersection of l.n. [r.n.] complexes is again a l. n. [r. n.] complex.)

By σ_H we denote the equivalence relation on S whose classes are the non-empty sets in the family of sets: H, $N(H)\setminus SN$, $\setminus N(H)$. If $\mathcal{F}$ is a non-empty family of non-empty subsets of S, we set $\sigma_{\mathcal{F}} = \bigcap_{H\in\mathcal{F}} \sigma_H$; if $\mathcal{F}$ is empty, $\sigma_{\mathcal{F}}$ denotes the universal relation on S. We let $\lambda_x = \sigma_{A(x)}$, $\varrho_x = \sigma_{B(x)}$, $\tau_x = \lambda_x \cap \varrho_x$. For $a\in S$, we write $H^{\cdot}.a = \{x\in S \mid xa\in H\}$, $H.^{\cdot}a = \{x\in S \mid ax\in H\}$.

Following [9], we say that a band S is *left (right) normal* if it satisfies the identity $xyz=xzy$ [$xyz=yxz$], *normal* if it satisfies the identity $xyzx=xzyx$. If ξ is a congruence on an arbitrary semigroup S such that S/ξ is a left normal, right normal, or normal band, respectively, ξ is called a *left normal, right normal, normal congruence* (l.n., r.n., n. congruence, respectively).

By S^0 we denote the semigroup S with zero adjoined (irrespective of whether S has a zero or not). U, L, R will, respectively, stand for a one element semigroup, two element left zero semigroup, two element right zero semigroup. If A is any set, $|A|$ denotes its cardinality.

If S_α, $\alpha\in A$, is a non-empty family of semigroups, $\prod_{\alpha\in A} S_\alpha$ denotes their *Cartesian (or direct) product*, that is, the semigroup defined on the Cartesian product of sets S_α with coordinatewise multiplication; if $A=\{1,2\}$ we write $S_1\times S_2$ instead of $\prod_{i=1}^{2} A_i$. S is a *subdirect product* of semigroups S_α if S isomorphic to a subsemigroup S' of $\prod_{\alpha\in A} S_\alpha$ such that for all $\alpha\in A$, $\pi_\alpha(S')=S_\alpha$ (π_α is the α-th projection). If $\{B_i\}_{i=1}^n$ is a partition of A and for every i, $1\leqq i\leqq n$, all semigroups S_α with $\alpha\in B_i$ are isomorphic to a semigroup T_i, we say that S is a subdirect product of $|B_1|$ copies of T_1, $|B_2|$ copies of T_2, ..., $|B_n|$ copies of T_n. Subdirect irreducibility is taken in the usual sense (a one element semigroup is excluded).

For all concepts and notation not mentioned above the reader is referred to [2]. We will omit all statements that can be obtained from our results by the left-right duality.

3. Basic properties of concepts used

We will repeatedly use the next proposition without express mention.

Proposition 1 (*cf.* [9], Theorem 10). *Any normal band S satisfies the identity*

$$ax_1x_2\dots x_nb = ax_{i_1}x_{i_2}\dots x_{i_n}b, \tag{1}$$

where $\{i_1, i_2, \dots, i_n\}$ is a permutation of the set $\{1, 2, \dots, n\}$.

Proof. We prove the case $n=2$; the general case is treated by induction. It is clear that normality implies (1) for $a=b$. Thus

$$axyb=(axyb)(axyb)=(axyba)(xyb)=(ayxba)(xyb)=$$
$$=(ayx)(baxyb)=(ayx)(bayxb)=(ayxb)(ayxb)=ayxb.$$

Theorem 1. *The intersection of a* l.n. *congruence and a* r.n. *congruence is a* n. *congruence. Conversely, every* n. *congruence* $\sim$ *is the intersection of the finest* l.n. *congruence on* S *containing* $\sim$ *and the finest* r.n. *congruence on* S *containing* $\sim$.

Proof. The first statement of the theorem is immediate. Hence let $\sim$ be a n. congruence and for any $x, y \in S$, define

$$x \overset{l}{\sim} y \quad \text{if and only if} \quad x \sim yx \quad \text{and} \quad y \sim xy,$$

$$x \overset{r}{\sim} y \quad \text{if and only if} \quad x \sim xy \quad \text{and} \quad y \sim yx.$$

We show that $\overset{l}{\sim}$ is the finest l.n. congruence on S containing $\sim$; the case of $\sim$ is treated analogously. If $x \overset{l}{\sim} y$ and $y \overset{l}{\sim} z$, then

$$x \sim yx \sim zyx \sim (zy)(zx) \sim y(zx) \sim (yz)x \sim zx$$

and analogously $z \sim xz$. Thus $x \overset{l}{\sim} z$, and $\overset{l}{\sim}$ is an equivalence relation (symmetry and transitivity are obvious). If $x \overset{l}{\sim} y$, then for any $z \in S$,

$$xz \sim yxz \sim (yz)(xz),$$

similarly $yz \sim (xz)(yz)$ so that $xz \overset{l}{\sim} yz$; analogously $zx \overset{l}{\sim} zy$ and hence $\overset{l}{\sim}$ is a congruence and is clearly a l.n. congruence. Let $\overset{l}{\approx}$ be any l.n. congruence containing $\sim$. Then for $x \overset{l}{\sim} y$, we have $x \sim yx$, $y \sim xy$ and thus $x \overset{l}{\approx} yx$, $y \overset{l}{\approx} xy$. Consequently

$$x \overset{l}{\approx} yx \overset{l}{\approx} y(xy) \overset{l}{\approx} yy \overset{l}{\approx} y,$$

that is, $\overset{l}{\sim}$ is contained in $\overset{l}{\approx}$. It follows easily that $\sim = \overset{l}{\sim} \cap \overset{r}{\sim}$.

The next theorem establishes a connection between l.n. complexes and l.n. congruences.

Theorem 2. *The following conditions on a complex* H *of* S *are equivalent:*

a) H *is a* l.n. *complex of* S;

b) σ_H *is a* l.n. *congruence on* S;

c) *for all* $a \in N(H)$, $H^{\cdot}.a = H$.

Proof. a) implies b). If $N(H) = S$, then H is a l.d.r.i. of S and $S/\sigma_H \cong L$, and if $N(H) = H$, then H is a face of S and σ_H is a semilattice congruence. Hence suppose that $H \neq N(H) \neq S$, and let $A = N(H) \setminus H$, $B = S \setminus N(H)$. Then by the definition of a l.n. complex, the following inclusions hold:

	H	A	B
H	H	H	B
A	A	A	B
B	B	B	B

where, e.g., $HA \subseteq H$, etc. Defining multiplication according to this table, we see that $\{H, A, B\} \cong L^0$ and thus σ_H is a l.n. congruence.

b) implies c). Since σ_H is a l.n. congruence and it has at most 3 classes, S/σ_H is isomorphic to one of the semigroups U, U^0, L, L^0. If $S/\sigma_H \cong U$, $H = S$; if $S/\sigma_H \cong U^0$, H is a face of S; if $S/\sigma_H \cong L$, H is a l.d.r.i. of S; in any of these cases,

c) is established without difficulty. Finally, if $S/\sigma_H \cong L^0$ and $a \in N(H)$, then c) follows easily from the above table since this case then reduces to considering the semigroup $\{H, A\}$ which is isomorphic to L.

c) implies a). This follows easily from the definition of a l.n. complex.

Proposition 2. *A complex H is a* l.n. *complex of S if and only if H is left dense in S and has the property: if $x \in S$ is such that every prime ideal of S which contains x also intersects H, then $Hx \subset H$.*

Proof. Necessity. Since H is left dense in $N(H)$, it is also left dense in S. If $x \in S$ has the property stated above, then $x \in N(H)$ since otherwise $S \setminus N(H)$ would be a prime ideal of S containing x and not intersecting H. The inclusion $Hx \subseteqq H$ now follows by the definition of a l.n. complex.

Sufficiency. As before, we conclude that $x \in S$ with the above property must be contained in $N(H)$. Thus H is a l.d.r.i. of $N(H)$ as desired.

Recall that $A(x)$ is the smallest l.n. complex of S containing x.

Theorem 3. *Let x be an element of S, let*

$$A_1(x) = x \cup xN(x),$$

$$A_{2n}(x) = \{y \in S \mid A_{2n-1}(x) \cap R(y) \neq \square\},$$

$$A_{2n+1}(x) = A_{2n}(x) \cup A_{2n}(x)N(A_{2n}(x)),$$

for $n = 1, 2, \ldots$. Then $A(x) = \bigcup_{n=1}^{\infty} A_n(x)$.

Proof. We write A_n and A instead of $A_n(x)$ and $A(x)$, respectively, and let $T = \bigcup_{n=1}^{\infty} A_n$. Since $a \in A$, we have $N(x) \subseteqq N(A)$ and thus $xN(x) \subseteqq AN(A) \subseteqq A$ whence $A_1 \subseteqq A$. Suppose that $A_n \subseteqq A$, $n \geqq 1$. If n is even, then $A_{n+1} = A_n \cup A_n N(A_n)$ and $A_n \subseteqq A$ implies $A_n N(A_n) \subseteqq AN(A) \subseteqq A$, that is, $A_{n+1} \subseteqq A$. If n is odd, then for $y \in A_{n+1}$ we have $A_n \cap R(y) \neq \square$. Thus $yz \in A_n$ for some $z \in S^1$; hence $yz \in A$ so that $y \in A$. Consequently $A_{n+1} \subseteqq A$ and by induction we conclude that $T \subseteqq A$.

For the opposite inclusion, it suffices to show that T is a l.n. complex. Let $yz \in T$; then $yz \in A_n$ for some n. We may suppose that n is even since $A_1 \subseteqq A_2 \subseteqq \ldots$. Then $A_{n-1} \cap R(yz) \neq \square$ whence $A_{n-1} \cap R(y) \neq \square$ and thus $y \in A_n \subseteqq T$. Next let $y \in T$ and $z \in N(T)$. Since $A_1 \subseteqq A_2 \subseteqq \ldots$, we have $N(A_1) \subseteqq N(A_2) \subseteqq \ldots$ and hence

$$z \in N(T) = N\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} N(A_n).$$

We thus have $y \in A_m$ and $z \in N(A_n)$ and we may suppose that $m = n$ and n is even. Hence $yz \in A_n N(A_n) \subseteqq A_{2n+1} \subseteqq T$. Consequently T is a l.n. complex of S and thus $A = T$.

Proposition 3. *Let H be a* n. *complex of S and let a be an element of S. Then*

a) $A(H) = H^{\cdot}.a$;

b) $H = (H^{\cdot}.a) \cap (H.^{\cdot}a)$;

c) $N(H) = N(H^{\cdot}.a)$.

Proof. Items a) and b) follow easily from the definitions and Theorem 2, [4]. Clearly $H^{\cdot}.a \subseteqq N(H)$, which implies $N(H^{\cdot}.a) \subseteqq N(H)$. Conversely, $H \subseteqq H^{\cdot}.a$ implies $N(H) \subseteqq N(H^{\cdot}.a)$ and c) is established.

Theorem 4. *The non-empty intersection H of a* l.n. *complex C and a* r. n. *complex D is a* n. *complex and $N(H)=N(C)\cap N(D)$. Conversely, if H is a* n. *complex, then $H=A(H)\cap B(H)$ and $N(H)=N(A(H))=N(B(H))$.*

Proof. Let C and D be as above and $H=C\cap D\neq\square$. Let $E=N(C)\cap N(D)$; $H=C\cap D\subseteqq E$ so that $N(H)\subseteqq E$ since E is then a face. Let $C'=C\cap E$ and $D'=D\cap E$; then

$$H=C\cap D=(C\cap N(C))\cap(D\cap N(D))=(C\cap E)\cap(D\cap E)=C'\cap D',$$

where $C'\subseteqq E$, $D'\subseteqq E$. Further, $C'E\subseteqq CE\subseteqq CN(C)\subseteqq C$ and $C'E\subseteqq E$ which implies $C'E\subseteqq C'$. Also $xy\in C'\subseteqq C$ implies $x\in C$ which together with $x\in E$ (since $xy\in C'\subseteqq E$ and E is a face) implies $x\in C'$. Consequently C' is a l.d.r.i. of E; similarly D' is a r.d.l.i. of E, and thus $H=C'\cap D'$ is a quasi dense subsemigroup of E. If $x\in E$, then for any $c\in C'$, $d\in D'$, $cxd\in C'\cap D'=H\subseteqq N(H)$ whence $x\in N(H)$. Thus $E\subseteqq N(H)$ the opposite inclusion being obvious, we have $E=N(H)$. Therefore $N(H)=N(C)\cap \cap N(D)$.

For the converse it suffices to apply Proposition 3.

Remark. It follows from the definitions that a complex H of S is a n. complex if and only if for all $x, y, z\in S$:

a) $x, y\in H$ implies $xy\in H$;

b) $x^2\in H$ implies $x\in H$;

c) $xyz\in H$ implies $xz\in H$;

d) $xz\in H$, $y\in N(H)$ implies $xyz\in H$.

4. Homomorphisms onto normal bands

Let S be a fixed semigroup, $\mathscr{A}$ the family of all proper l.n. complexes of S together with the empty set, and $\mathfrak{A}$ the set of all l.n. congruences on S. On the set $\mathfrak{P}(\mathscr{A})$ of all non-empty subsets of $\mathscr{A}$ define the function α by: $\alpha(\mathscr{A}')=\sigma_{\mathscr{A}'}$ (for notation see section 2). Then we have the following result which is fundamental for most of this paper.

Theorem 5. *The function α maps $\mathfrak{P}(\mathscr{A})$ onto $\mathfrak{A}$ and is inclusion inverting.*

Proof. If $\mathscr{A}'=\square$, $\sigma_{\mathscr{A}'}$ is the universal relation and hence $\sigma_{\mathscr{A}'}\in\mathfrak{A}$. Otherwise $\mathscr{A}'\neq\square$ which implies that for every $A\in\mathscr{A}'$, σ_A is a l.n. congruence by Theorem 2; consequently $\sigma_{\mathscr{A}'}=\bigcap_{A\in\mathscr{A}'}\sigma_A\in\mathfrak{A}$. This shows that α maps $\mathfrak{P}(\mathscr{A})$ into $\mathfrak{A}$.

We show next that α maps $\mathfrak{P}(\mathscr{A})$ onto $\mathfrak{A}$. Hence let $\sim$ be any proper l.n. congruence on S. For every $x\in S$, let

$$A_x=\{y\in S \mid x\sim yx\},$$

and let $\mathscr{A}'$ be the family of all distinct sets A_x such that $A_x\neq S$, as x ranges over all elements of S. We will show that $\mathscr{A}'\in\mathfrak{P}(\mathscr{A})$ and that $\sigma_{\mathscr{A}'}=\sim$.

$\mathscr{A}'$ is not empty for in such a case we would have $A_x = S$ for all $x \in S$ which would imply for all $x, y \in S$, $x \sim yx$, $y \sim xy$. But then

$$x \sim yx \sim xyx \sim xy \sim y$$

contradicting the hypothesis that $\sim$ is proper. Thus to show that $\mathscr{A}' \in \mathfrak{P}(\mathscr{A})$, it suffices to prove that for all $x \in S$, $A_x \in \mathscr{A}$ if $A_x \neq S$. We fix $x \in S$ and let

$$T_x = \{y \in S \mid x \sim xy\}.$$

If $y, z \in T_x$, then $x \sim xy \sim xz$ and thus $x \sim (xy)(xz) \sim xyz$, that is, $yz \in T_x$. Conversely, if $yz \in T_x$, then $x \sim xyz$ and hence

$$x \sim xyz \sim xxyz \sim (xy)(xyz) \sim xyx \sim xxy \sim xy,$$

that is, $y \in T_x$; similarly $z \in T_x$. Consequently T_x is a face of S. If $y \in A_x$, then $x \sim yx$ whence $x \sim xyx \sim xy$ and thus $y \in T_x$. Hence $A_x \subseteqq T_x$ and thus $N(A_x) \subseteqq T_x$ since T_x is a face. Further, if $y \in T_x$, then $x \sim xy$ whence $x \sim (xy)x$ so that $xy \in A_x \subseteqq N(A_x)$. But $xy \in N(A_x)$ implies $y \in N(A_x)$ which proves $T_x \subseteqq N(A_x)$. Consequently $T_x = N(A_x)$.

If $yz \in A_x$, then $x \sim yzx$ and we have $yx \sim y(yzx) \sim yzx \sim x$ so that $y \in A_x$. If $y \in A_x$, $z \in T_x$, then $x \sim yx \sim xz$ and thus $x \sim yx \sim yxz \sim (yz)x$, that is, $yz \in A_x$. Consequently A_x is a l.d.r.i. of $T_x = N(A_x)$ and hence $A_x \in \mathscr{A}$ if $A_x \neq S$.

We show next that $\sigma_{\mathscr{A}'} = \sim$. Suppose that $x \sigma_{\mathscr{A}'} y$. Then $x \in A_x$ implies that $y \in A_x$, that is, $x \sim yx$; dually $y \sim xy$ and thus $x \sim yx \sim xyx \sim xy \sim y$.

Conversely, suppose that $x \sim y$ and let $z \in S$. If $x \in A_z$, then $z \sim xz$ and thus $z \sim yz$, that is, $y \in A_z$. If $x \in N(A_z) \setminus A_z$, then $z \sim zx$ since $x \in N(A_z) = T_z$, which implies $z \sim zy$ and $y \in N(A_z)$. If y were an element of A_z, then $z \sim yx$ which would imply $z \sim yz \sim xz$, that is, $x \in A_z$ contradicting the hypothesis. Consequently $y \in N(A_z) \setminus T_z$. The implications established also prove that $x \notin N(A_z)$ implies $y \notin N(A_z)$. By symmetry we conclude that $x \sigma_{A_z} y$ and since z is arbitrary, also $x \sigma_{\mathscr{A}'} y$. Therefore $\sigma_{\mathscr{A}'} = \sim$.

The last statement of the theorem is now clear.

Corollary. *$\sigma_{\mathscr{A}}$ is the finest* l.n. *congruence on* S.

Let $\mathscr{C}$ be the family of all proper l.n. complexes and proper r.n. complexes of S together with the empty set, and $\mathfrak{C}$ be the set of all l.n. congruences and r.n. congruences on S. On the set $\mathfrak{P}(\mathscr{C})$ of all non-empty subsets of $\mathscr{C}$ define the function γ by: $\gamma(\mathscr{C}') = \sigma_{\mathscr{C}'}$.

Theorem 6. *The function γ maps $\mathfrak{P}(\mathscr{C})$ onto $\mathfrak{C}$ and is inclusion inverting.*

Proof. This follows easily from Theorem 5 and its dual, and Theorem 1.

Corollary 1. *$\sigma_{\mathscr{C}}$ is the finest* n. *congruence on* S.

Letting $\mathscr{D}$ be the family of all n. complexes of S, we have the following result by the preceding corollary and Theorem 1.

Corollary 2. *$\sigma_{\mathscr{D}}$ is the finest* n. *congruence on* S.

Note that $\mathscr{C}$ can not be replaced by $\mathscr{D}$ in Theorem 6.

Recall that for any $x \in S$, $\lambda_x = \sigma_{A(x)}$, $\varrho_x = \sigma_{B(x)}$, $\tau_x = \lambda_x \cap \varrho_x$. Hence by Theorem 2 (its dual) $\lambda_x [\varrho_x]$ is a l.n. [r.n.] congruence and hence by Theorem 1, τ_x is a n. congruence. It is not hard to show that $\sigma_{\mathscr{A}} = \bigcap_{x \in S} \lambda_x$ and $\sigma_{\mathscr{Q}} = \sigma_{\mathscr{C}} = \bigcap_{x \in S} \tau_x$. The next proposition follows easily from the definitions.

Proposition 4. *For any $x \in S$, we have*

a) $S/\lambda_x \cong U$ *if and only if* $A(x) = S$;

b) $S/\lambda_x \cong L$ *if and only if* $A(x) \neq N(x) = S$;

c) $S/\lambda_x \cong U^0$ *if and only if* $A(x) = N(x) \neq S$;

d) $S/\lambda_x \cong L^0$ *if and only if* $A(x) \neq N(x) \neq S$.

The next theorem characterizes the n. congruences τ_x.

Theorem 7. *For any $x \in S$, we have:*

a) $S/\tau_x \cong U$ *if and only if* $A(x) = S = B(x)$;

b) $S/\tau_x \cong L$ *if and only if* $A(x) \neq S = B(x)$;

c) $S/\tau_x \cong R$ *if and only if* $A(x) = S \neq B(x)$;

d) $S/\tau_x \cong L \times R$ *if and only if* $A(x) \neq N(x) = S \neq B(x)$;

e) $S/\tau_x \cong U^0$ *if and only if* $A(x) = B(x) \neq S$;

f) $S/\tau_x \cong L^0$ *if and only if* $A(x) \neq N(x) = B(x) \neq S$;

g) $S/\tau_x \cong R^0$ *if and only if* $B(x) \neq N(x) = A(x) \neq S$;

h) $S/\tau_x \cong (L \times R)^0$ *if and only if* $A(x) \neq N(x) \neq B(x)$, $N(x) \neq S$,

and these are all homomorphic images S/τ_x.

Proof. As a sample we outline the proof of h); the other cases are treated analogously (most of them are simpler to prove than h)). We note first that by Theorem 4, $N(x) = N(A(x)) = N(B(x))$.

Necessity of h). Since S/τ_x has a zero, we must have $N(x) \neq S$. Then $N(x)/\xi_x \cong$ $\cong L \times R$, where ξ_x is the restriction of τ_x to $N(x)$, which by d) (d) in turn follows easily from Proposition 4, part b) and its dual) implies that $A(x) \neq N(x) \neq B(x)$. Sufficiency of h) is proved by essentially reversing the steps in the proof of necessity.

The last statement of the theorem follows by enumerating all possible cases of relationships among $A(x)$, $B(x)$, $N(x)$, and S.

5. Some properties of normal bands

We now investigate the properties of l.n. [r.n.] complexes of normal bands. The next theorem will be useful, it is also of independent interest.

Theorem 8. *Let S be any semigroup such that for all $a, x, y, b \in S$, $axyb = ayxb$. Then for any $x \in S$, we have:*

a) $N(x) = \{y \in S \mid SyS \cap \langle x \rangle \neq \Box\}$ *(the smallest face containing x)*;

b) $P(x)=\{y\in S|\ yS\cap xS\neq\Box\}$ *(the smallest* l.d.r.i. *containing* x*)*;

c) $A(x)=\{y\in S|\ yS\cap\langle x\rangle\neq\Box\}$ *(the smallest* l.n. *complex containing* x*)*.

Proof. a) Let $T(x)=\{y\in S|\ SyS\cap\langle x\rangle\neq\Box\}$. If $a,b\in T(x)$, then $uav=x^m$, $zbw=x^n$ for some $u,v,z,w\in S$ and some m,n. Hence $(uz)ab(vw)=(uav)(zbw)=$ $=x^{m+n}$ and thus $ab\in T(x)$. Conversely, if $ab\in T(x)$, then $uabv=x^m$ for some $u,v\in S$ and some m, which implies $a,b\in T(x)$. Thus $T(x)$ is a face of S containing x and hence $N(x)\subseteq T(x)$. The opposite inclusion being evident, we have $N(x)=T(x)$.

b) Let $T(x)=\{y\in S|\ yS\cap xS\neq\Box\}$. If $a\in T(x)$, $b\in S$, then $au=xv$ for some $u,v\in S$. Thus $aub^2=xvb^2$ whence $ab(ub)=x(vb^2)$ so that $ab\in T(x)$. Conversely, if $ab\in T(x)$, then obviously $a\in T(x)$. Hence $T(x)$ is a l.d.r.i. of S containing x and thus $P(x)\subseteq T(x)$. Again the opposite inclusion is obvious, and we have $P(x)=T(x)$.

c) Let $T(x)=\{y\in S|\ yS\cap\langle x\rangle\neq\Box\}$. By the definition of $A(x)$ and parts a), b) of the present theorem, $A(x)=\{y\in N(x)|\ yN(x)\cap xN(x)\neq\Box\}$. If $a\in T(x)$, then $au=x^n$ for some $u\in S$ and some n. Hence $a(ux)=xx^n$, where clearly $ux, x^n\in N(x)$ and $a\in A(x)$. Conversely, suppose that $a\in A(x)$. Then $a\in N(x)$ and $au=xv$ for some $u,v\in N(x)$. By part a), $zvw=x^m$ for some $z,w\in S$ and some m. Consequently $a(uzw)=xvzw=x(zvw)=x^{m+1}$, and $a\in T(x)$. Therefore $A(x)=T(x)$.

Corollary 1. *In a normal band* S, *for any* $x\in S$, *we have:*

a) $N(x)=\{y\in S|\ x=xyx\}$;

b) $P(x)=\{y\in S|\ ya=xa \text{ for some } a\in S\}$;

c) $A(x)=\{y\in S|\ x=yx\}$.

Proof. a) This is valid in any band S (6.2, [3]).

b) If $yu=xv$, then $y(vu)=(yu)(vu)=(xv)(vu)=x(vu)$.

c) If $yu=x$, then $yx=yu=x$.

Corollary 2. *In a normal band* S, *the following statements are equivalent for any* $x\in S$:

a) N_x *is a left zero semigroup*;

b) x *is a left zero of* $N(x)$;

c) $B(x)=N(x)$.

Proof. We prove only that c) implies a). If $a\in N_x$, then $x=xax$ by part a) of Corollary 1 which together with the hypothesis and the dual of part c) of Corollary 1 implies $x=xa$.

Note that a) and b) in Corollary 2 are equivalent in any band. Let S be a normal band, and let

$$D_1=\{x\in S|\ |N_x|=1\},$$
$$D_2=\{x\in S|\ |N_x|>1 \text{ and } N_x \text{ is a left zero semigroup}\},$$
$$D_3=\{x\in S|\ |N_x|>1 \text{ and } N_x \text{ is a right zero semigroup}\},$$
$$D_4=S\setminus(D_1\cup D_2\cup D_3).$$

Further, let K be the intersection of all ideals of S and let

$$F_i = D_i \cap K,$$
$$G_i = D_i \cap (S \setminus K),$$

for $i = 1, 2, 3, 4$. Finally let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ denote, respectively, the semigroups U, L, R, $L \times R$. We are now ready to prove the main theorem of this section.

Theorem 9. *Let x be any element of a normal band S. Then for $i = 1, 2, 3, 4$,*

a) $S/\tau_x \cong \alpha_i$ *if and only if* $x \in F_i$,

b) $S/\tau_x \cong \alpha_i^0$ *if and only if* $x \in G_i$.

Proof. We note that $K = \{y \in S \mid N(y) = S\}$, that by Theorem 4, $N(x) = N(A(x)) = N(B(x))$, and that $A(x) = B(x)$ implies $A(x) = N(x)$. The theorem now follows from Theorem 7 and Corollary 2 to Theorem 8 and its dual.

We say that a family $\mathscr{F}$ of equivalence relations on a set T distinguishes elements of T if $\bigcap_{\varphi \in \mathscr{F}} \varphi = \iota_T$, where ι_T is the identity relation on T.

Theorem 10. *S is a normal band if and only if the family $\{\tau_x\}_{x \in S}$ distinguishes elements of S.*

Proof. Necessity. Let $x, y \in S$ and suppose that for all $z \in S$, $x \tau_z y$. In particular $x \tau_x y$ which implies $y \in A(x) \cap B(x)$. By minimality of $A(x)$ and $B(x)$, we have $A(y) \subseteqq A(x)$ and $B(y) \subseteqq B(x)$. Similarly $x \tau_y y$ implies $A(x) \subseteqq A(y)$, $B(x) \subseteqq B(y)$. Consequently $A(x) = A(y)$, $B(x) = B(y)$. By Corollary 1 to Theorem 8 (part c) and its dual), we have for all $z, w, \in S$:

$$x = zx \quad \text{if and only if} \quad y = zy,$$

$$x = xw \quad \text{if and only if} \quad y = yw.$$

Since x and y are idempotent, $x = xy = y$.

Sufficiency. Since τ_x is a n. congruence, so is $\bigcap_{x \in S} \tau_x$ which by the hypothesis is equal to ι_S, the identity relation on S. But then S itself is a normal band.

One similarly establishes

Proposition 5. *S is a left normal band if and only if the family $\{\lambda_x\}_{x \in S}$ distinguishes elements of S.*

6. Normal bands and subdirect products

We next use some results of the previous section to obtain representations of normal bands as subdirect products of subdirectly irreducible normal bands.

Let S be a normal band. Then by Theorem 10, $\bigcap_{x \in S} \tau_x = \iota_S$ which by BIRKHOFF's theorem (Theorem 9, p. 92, [1]) implies that S is a subdirect product of semigroups S/τ_x, $x \in S$. Theorem 9 yields all the semigroups S/τ_x in terms of the sets F_i and G_i, $i = 1, 2, 3, 4$. In the notation of that theorem, $S/\tau_x \cong \alpha_1 = U$ if and only if $x \in F_1$.

Since $F_1 = D_1 \cap K$ it is clear that if $F_1 \neq \Box$, F_1 consists of a single element which is the zero of S. Further, if $S/\tau_x \cong \alpha_4^0 = (L \times R)^0$, then by the definition of τ_x ($\tau_x = \lambda_x \cap \varrho_x$), we must have $S/\lambda_x \cong L^0$, $S/\varrho_x \cong R^0$. It is easy to verify that U^0, L, R, L^0, R^0 are subdirectly irreducible. Using Theorem 9 for counting the number of copies of these semigroups among the semigroups S/τ_x and taking into account the above discussion, we obtain the next theorem which is the main result of this section.

Theorem 11. *Any normal band S having more than one element is the subdirect product of*

$$|F_2| + |F_4| \ \textit{copies of} \ L, \qquad |F_3| + |F_4| \ \textit{copies of} \ R,$$

$$|G_1| \ \textit{copies of} \ U^0, \quad |G_2| + |G_4| \ \textit{copies of} \ L^0, \qquad |G_3| + |G_4| \ \textit{copies of} \ R^0.$$

The total number of copies is $|S|$ if S has no zero and $|S|-1$ if S has a zero. The semigroups L, R, U^0, L^0, R^0 are subdirectly irreducible.

Remark. We have already noted that if $F_1 \neq \Box$, then it consists of a single element which is the zero of S. Similarly Corollary 2 to Theorem 8 implies that if $F_2 \neq \Box$, then F_2 is the set of all left zeros of S and is the kernel of S; an analogous statement is valid for F_3. Hence at most one of the sets F_1, F_2, F_3 is non-empty.

Corollary 1. *All the semigroups considered contain more than one element.*

a) *A left normal band is the subdirect product of*

$$|F_2| \ \textit{copies of} \ L, \qquad |G_1| \ \textit{copies of} \ U^0, \qquad |G_2| \ \textit{copies of} \ L^0.$$

b) *A left zero semigroup is a subdirect product of*

$$|S| \ \textit{copies of} \ L.$$

c) *A semilattice is the subdirect product of*

$$|S|-1 \ \textit{copies of} \ U^0 \ \textit{if} \ S \ \textit{is finite},$$

$$|S| \ \textit{copies of} \ U^0 \ \textit{if} \ S \ \textit{is infinite}.$$

Proof. Part a) follows from the theorem; it can also be derived directly, by the same method of proof as above, from Propositions 4 and 5. Part b) follows directly from the theorem. If S is a finite semilattice, then S has a zero. If S is an infinite semilattice with zero, then $|S| = |S \setminus 0| = |G_1|$. Hence c) holds.

Corollary 2 (*cf.* Theorem 1, [5] and Corollaire I, Théorème V, [7]). *These are the only subdirectly irreducible*

a) *normal bands:* L, R, U^0, L^0, R^0;

b) *left normal bands:* L, U^0, L^0;

c) *left zero semigroups:* L;

d) *semilattices:* U^0.

Corollary 3. (*cf.* Theorem 4, [9] which is a stronger statement). *Every normal band is a subdirect product of a left and a right normal band.*

Proof. This follows from the theorem since the product of, e.g., copies of L, U^0, L^0 is a left normal band; similarly for R, U^0, L^0. This corollary also follows from Theorem 1.

One might ask what kind of bands are subdirect products of, say, left zero semigroups and semilattices, or some other combination of classes of semigroups we have considered. The desired results can be obtained from Theorem 11 or certain of its corollaries.

7. A representation of normal bands

The following construction is an easy modification of the one given by Shain [5] for right normal bands. It gives a representation of a normal band by subsets of a set under certain multiplication.

Let B, C, and D be sets, let $E=\{1, 2\}$ and suppose that B, $C\times E$, $D\times E$ are pairwise disjoint. Let

$$A=B\cup(C\times E)\cup(D\times E),$$

and let $\mathcal{F}$ be the set of all subsets of A under the following multiplication: for $\mathfrak{A}, \mathfrak{B}\in\mathcal{F}$;

$$\mathfrak{A}\cdot\mathfrak{B}\cap B=\mathfrak{A}\cap\mathfrak{B}\cap B, \tag{1}$$

$$\mathfrak{A}\cdot\mathfrak{B}\cap(C\times E)=\mathfrak{A}\cap(C\times E),$$

$$\mathfrak{A}\cdot\mathfrak{B}\cap(D\times E)=\mathfrak{B}\cap(D\times E).$$

It is easy to see that $\mathcal{F}$ is a normal band; in analogy to [5] we call subsemigroups of $\mathcal{F}$ *special normal bands.*

Theorem 12. *Every normal band is isomorphic to a special normal band.*

Proof. Let S be a normal band; then using Theorem 11, we may suppose that $S\subseteq\prod_{i\in I} S_i$, where S_i is one of the semigroups: L, R, U^0, L^0, R^0, and projection $\pi_i(S)=S_i$ for all $i\in I$. Let

$$B=\{i\in I\mid S_i=U^0\},$$

$$C=\{i\in I\mid S_i=L \quad\text{or}\quad S_i=L^0\},$$

$$D=\{i\in I\mid S_i=R \quad\text{or}\quad S_i=R^0\}.$$

Let $U=\{1\}$, $L=\{l_1, l_2\}$, $R=\{r_1, r_2\}$. Let A, E, and $\mathcal{F}$ be as above and define the function φ: $S\to\mathcal{F}$ by letting $\varphi(x)\subseteq A$ be defined by:

$$\varphi(x)\cap B = \{i\in B\mid \pi_i(x) = 1\},$$

$$\varphi(x)\cap(C\times E) = \bigcup_{j=1}^{2} \{(i,j)\in C\times E\mid \pi_i(x) = l_j\}, \tag{2}$$

$$\varphi(x)\cap(D\times E) = \bigcup_{j=1}^{2} \{(i,j)\in D\times E\mid \pi_i(x) = r_j\}.$$

A straightforward calculation shows that φ is an isomorphism; here $\varphi(S)$ is a special normal band.

Theorem 13 (*cf.* Theorem 2, [5] and Corollaire II, Théorème V, [7]). *With the notation as in the introduction to this section, we have:*

a) $\mathscr{F}$ *is a left normal band if and only if* $D = \square$;

b) $\mathscr{F}$ *is a left zero semigroup if and only if* $B = D = \square$;

c) $\mathscr{F}$ *is a semilattice if and only if* $C = D = \square$;

d) $\mathscr{F}$ *is a rectangular band if and only if* $B = \square$.

Defining a "special left normal band", a "special left zero semigroup", etc., analogously as a special normal band above, the statements corresponding to Theorem 12 *are valid for bands in* a)—d).

Proof. The proof is an adaptation of the proof of Theorem 12 and is omitted.

Remark. If we replace (1) by

$$\mathfrak{A}\cdot\mathfrak{B} \cap B = (\mathfrak{A} \cup \mathfrak{B}) \cap B,$$

$\mathscr{F}$ is still a normal band and Theorem 12 remains valid where in the proof, (2) is replaced by

$$\varphi(x) \cap B = \{i \in B \mid \pi_i(x) = 0\}$$

(0 is the zero of U^0).

Bibliography

[1] G. Birkhoff, *Lattice theory* (New York, 1948).
[2] A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*. Vol. I (Providence, 1961).
[3] M. Petrich, The maximal semilattice decomposition of a semigroup, *Math. Z.*, **85** (1964), 68—82.
[4] M. Petrich, The maximal matrix decomposition of a semigroup, *Portugaliae Math.* (to appear).
[5] B. M. Shain, A contribution to the theory of restrictive semigroups, *Izv. vys. uch. zav., Mat.*, **2** (33) (1963), 152—154 (Russian).
[6] B. M. Shain, Restrictive bisemigroups, *Izv. vys. uch. zav., Mat.*, **1** (44) (1965), 168—179 (Russian).
[7] G. Thierrin, Sur la structure des demi-groupes, *Alger Math.*, **3** (1956), 161—171.
[8] V. V. Vagner, Restrictive semigroups, *Izv. vys. uch. zav., Mat.*, **6** (31) (1962), 19—27 (Russian).
[9] M. Yamada and N. Kimura, Note on idempotent semigroups. II, *Proc. Japan Acad.*, **34** (1958), 110—112.

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(Received August 12, 1965)

On some results of A. Rényi and C. Rényi concerning periodic entire functions

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In their paper [7] A. and C. RÉNYI have investigated the possibility that $f(g(z))$ should be an entire periodic function, where $f(z)$ and $g(z)$ are entire and $g(z)$ is non-periodic. They proved the following two theorems:

Theorem 1. *If $f(z)$ is an arbitrary non-constant entire function and $P(z)$ an arbitrary polynomial of degree $\geqq 3$, then the entire function $f(P(z))$ is not periodic.*

Theorem 2. *If $P(z)$ is an arbitrary non-constant polynomial and $g(z)$ an entire function which is not periodic, then $P(g(z))$ is not periodic.*

In this note we shall make some improvements on theorem 2 and prove first

Theorem 3. *If $f(z)$ is an arbitrary non-constant entire function of order less than $\frac{1}{2}$ or of order $\frac{1}{2}$ and minimal type, and if $g(z)$ is an entire function which is not periodic, then $f(g(z))$ is not periodic.*

The proof depends on the following

Lemma 1. *If $f(z)$ and $g(z)$ are non-constant entire functions such that $g(z)$ is not periodic but $f(g(z))=F(z)$ is periodic with period λ, and if t is a value for which $F'(t)\neq 0$, then for integral n, $w_n=g(t+n\lambda)$ satisfies $f(w_n)=F(t)$, while $w_n=w_m$ if and only if $n=m$.*

Consequently $|g(t+n\lambda)|\to\infty$ as $|n|\to\infty$ and $g(z)$ is unbounded on any ray $z=z_0+\lambda s$, $0\leqq s<\infty$.

Proof of Lemma 1. It is clear that w_n satisfies $f(w_n)=f(g(t+n\lambda))= = F(t+n\lambda)=F(t)=a$, say. If there are integers m, n such that $m\neq n$, $w_m=w_n$, then consideration of

$$a(s)=F(t+m\lambda+s)=f(g(t+m\lambda+s))=F(t+n\lambda+s)=f(g(t+n\lambda+s))$$

shows that for all sufficiently small $|s|$,

$$g(t+m\lambda+s)=g(t+n\lambda+s)$$

is the unique solution of $f(w)=a(s)$ near $w_m=w_n$ (we recall that $f'(w_m)\neq 0$ since $f'(w_m)g'(t+m\lambda)=F'(t+m\lambda)=F'(t)\neq 0$). It then follows that $g(z)$ is periodic with period $(m-n)\lambda$, against assumption. Thus we conclude that $w_m\neq w_n$ for $m\neq n$.

Since w_n is a solution of $f(w_n)=a$, it follows that $|w_n|\to\infty$ as $|n|\to\infty$. On any ray L: $z=z_0+\lambda s$, $s\geqq 0$, one can find $z=t$ such that $F'(t)\neq 0$, and the $t+n\lambda$ then lie on L so that $g(z)$ is unbounded on L.

Proof of theorem 3. Suppose that f and g satisfy the conditions of theorem 3, but that $F(z)=f(g(z))$ is periodic with period, say, λ. Denote by L any ray of the form $z=z_0+\lambda s$, $s\geqq 0$. Now $g(L)$ is an unbounded, connected plane set on which $f(z)$ is bounded, since the values of $f(z)$ on $g(L)$ are the values of $F(z)=f(g(z))$ on L, and these are bounded, being in fact the values taken on the intersection of L with one period strip of F. But (see e.g. [3]) for a function of the order of growth of $f(z)$ there is a sequence $R_n\to\infty$ such that the minimum of $|f(z)|$ tends to ∞ as $|z|\to\infty$ through the values R_n. This contradicts the boundedness of $f(z)$ on $g(L)$. Thus the original assumption that $f(g(z))$ is periodic is false.

Theorem 3 is sharp, since to a prescribed type $\varepsilon>0$ the function $f(z)=\cosh(\varepsilon\sqrt{z})$ is of order $\frac{1}{2}$, type ε and $f(z^2)=\cosh(\varepsilon z)$ is periodic. We may note, however, that in this example the "inner" function $g(z)$ is the polynomial z^2 and it is natural to see if more can be proved when the case of quadratic $g(z)$ is set aside. Concerning this case one can at least prove

Theorem 4. *If $f(z)$ is an arbitrary non-constant entire function, if $g(z)$ is a non-periodic entire function other than a polynomial of degree $\leqq 2$, and if $F(z)=f(g(z))$ is periodic, then*

(i) *$g(z)$ is transcendental, and* (ii) *the order of $F(z)$ is infinite.*

Proof. Part (i) follows at once from Theorem 1. The proof of part (ii) follows at once from the result of Pólya [6] that if $F=f(g)$ is of finite order, where f, g are entire, then either g is a polynomial and f is of finite order or g is not a polynomial and f is of zero order. Since in our case g is not a polynomial and f has order $\geqq\frac{1}{2}$ we conclude that $F(z)$ has infinite order.

For the further discussion we note that without loss of generality z may be subjected to a linear transformation to make the period of $F(z)$ equal to $2\pi i$, i.e. $F(z)$ may be represented as $h(e^z)$, where $h(z)=\sum_{-\infty}^{\infty}A_n z^n$, $0<|z|<\infty$. The case when $h(z)$ is entire, i.e. when $A_n=0$ for negative n, may be distinguished as the case when $F=h(e^z)$ is bounded in the left half-plane $\operatorname{Re} z<0$. In this case it is impossible that a non-constant $F=f(g(z))$, where $g(z)$ is a quadratic polynomial, for such a decomposition together with the boundedness of F in a left half-plane would imply boundedness in a right half-plane also and thus F would be a (periodic) entire function bounded in the whole plane. We can prove rather more:

Theorem 5. *If $h(z)$ is a non-constant entire function and $F(z)=h(e^z)$, and if F may also be represented as $F=f(g(z))$, where $g(z)$ is a non-periodic entire function and $f(z)$ is an entire function, then*

(i) *the order of $f(z)$ is at least one,*
(ii) *the order of $g(z)$ is at least one,*
(iii) *$g(z)$ is p-valent in a suitable left half-plane H:* $\operatorname{Re} z<\text{const.}$ *for some integer p, i.e. g takes any value at most p times in H,*
(iv) *the order of $F(z)$ is infinite, so that $h(z)$ must be transcendental.*

In the course of this proof we shall need two lemmas:

Lemma 2. *If for a positive integer p an entire function $g(z)$ is p-valent in a half-plane H:* $\operatorname{Re} z < C = \text{const.}$ *in the sense that $g(z)$ takes no value more than p times in H, then*

$$|g(z)| = O(|z|^{2p})$$

uniformly as $z \to \infty$ in any angle A: $|\arg z - \pi| \leqq \theta$, where θ is a constant less than $\pi/2$.

Such a result was proved by BIEBERBACH [2] in the case $p = 1$ of univalent functions.

Proof of Lemma 2. Put $t = z - C$, $s = (t+1)/(t-1)$. This substitution maps H one-to-one conformally on D: $|s| < 1$ in such a way that $z = \infty$ corresponds to $t = 1$. Moreover $\varphi(s) \equiv g(z)$ is p-valent in D. By Miss CARTWRIGHT's results [4] one has

$$|\varphi(s)| = O\{(1-|s|)^{-2p}\} \quad \text{uniformly as} \quad |s| \to 1 \quad \text{in } D.$$

Now, as $z \to \infty$ in an angle A, so that for large $|z|$, z is in H, we have

$$1 - |s|^2 = 1 - \frac{t+1}{t-1} \cdot \frac{\bar t + 1}{\bar t - 1} = -2 \operatorname{Re} t/(t \bar t - t - \bar t + 1)$$

$$> -2 \operatorname{Re} t/(|t|+1)^2 \quad \text{since} \quad \operatorname{Re} t < 0,$$

$$> K|z| \cos \theta / |z|^2 = K'/|z| \quad \text{for a suitable constant } K'.$$

Thus $|g(z)| = |\varphi(s)| = O\{(1-|s|)^{-2p}\} = O\{(1-|s|^2)^{-2p}\} = O(|z|^{2p})$.

The next lemma is essentially the Phragmén—Lindelöf principle in a form convenient for our application. It is proved in this form in e.g. [1].

Lemma 3. *If the order of the entire function $f(z)$ is $\leqq \beta$, $\beta > 0$, and if as $z \to \infty$ outside a number of disjoint angular sectors of the form D:*

$$\theta_1 < \arg z < \theta_2, \quad \theta_2 - \theta_1 < \frac{\pi}{\beta},$$

one has

$$|f(z)| = O\left(\exp\left(|z|^{\beta'}\right)\right), \ \beta' < \beta, \ K \ \textit{constant},$$

then the order of $f(z)$ is in fact $\leqq \beta'$.

Taking $\beta < 1$ one immediately obtains the

Corollary. *An entire function of order $\beta < 1$ cannot be of order strictly $< \beta$ (in particular cannot be $O(|z|^k)$) in any half-plane.*

Proof of theorem 5. If $h(0) = \alpha$, then $F(z) \to \alpha$ as $\operatorname{Re} z \to -\infty$.
First we show $|g(x)| \to \infty$ as $x \to -\infty$. Suppose this is not true. Then there is a $K > 0$ and a sequence $x_n \to -\infty$, such that $|g(x_n)| \leqq K$. Now $F(x) = f(g(x)) \to \alpha$ as $x \to -\infty$. We can assume, by choosing a subsequence of x_n if necessary, that $g(x_n) \to \beta$, $|\beta| \leqq K$. Then $f(\beta) = \alpha$. Now given $\varepsilon > 0$, we have $|f(g(x)) - \alpha| < \varepsilon$ for all sufficiently large $-x$, and since the α-points of f are isolated, it follows that $g(x) \to \beta$ as $x \to -\infty$. Indeed $f(g(z)) \to \alpha$ as $\operatorname{Re} z \to -\infty$, so that we must have $g(z) \to \beta$ as

$\operatorname{Re} z \to -\infty$. But this is impossible, since by Lemma 1 $g(z)$ is unbounded on any line $\operatorname{Re} z = \text{constant}$. Thus we have shown that $|g(x)| \to \infty$ as $x \to -\infty$. Consider the half-plane H: $\operatorname{Re} z < c$, where c is chosen so that $h(z)$ takes the value α in $|z| < e^c$ only at $z=0$, while $|h(z)-\alpha| > \delta > 0$ on $|z| = e^c$. Then in H one has $F(z) \neq \alpha$, while on the boundary of H one has $|F(z)-\alpha| > \delta > 0$. As $z \to \infty$ on the negative real axis R in H one has $F(z) \to \alpha$.

The values of $w = g(z)$ in H form an unbounded domain $G = g(H)$ and as $z \to \infty$ along R in H, the corresponding values $g(z)$ run to ∞ along a path L in G. As $w \to \infty$ along L one has $f(w) \to \alpha$. Moreover $f(w) \neq \alpha$ for w in G, while at any boundary point of G, $|f(w)-\alpha| > \delta$. Consequently (see e.g. [5, Chapter XI]) α is a direct critical transcendental singularity of the inverse function $f_{-1}(z)$ of $f(w)$. By the Denjoy—Carleman—Ahlfors theorem [5, p. 313] it follows that the order of $f(z)$ is at least one. This proves part (i).

Let the integer p denote the multiplicity of $z=0$ as a solution of $h(z)=\alpha$. Then the number $\varepsilon > 0$ and the c in the definition of H may be chosen so that, for $0 < |\alpha' - \alpha| < \varepsilon$ the equation $h(t) = \alpha'$ has exactly p roots, all different, $t_1, t_2, \ldots, t_p$ in $|t| < e^c$. Thus in H we have $F(z) = h(e^z) = \alpha'$ precisely at the infinite set of points

$$S(\alpha') = \{\log t_1, \log t_2, \ldots, \log t_p\}.$$

We may assume that $F'(z) \neq 0$ in H. Then by Lemma 1 $g(z)$ does not take the same value at any two different values of $\log t_i$ (for fixed i). Then on the set $S(\alpha)$ the function $g(z)$ can take a given value at most p times, i.e. once on a value of $\log t_1$, once on a value of $\log t_2$, etc.

Now if $g(z) = g(z')$ for z, z' in H, then $F(z) = F(z')$ so that z and z' belong to the same set $S(\alpha')$. Hence we have proved (iii) that $g(z)$ takes any given value at most p times in H.

By Lemma 2 we see that $g(z)$ is $O(|z|^{2p})$ in H and it follows from Lemma 3 that the order of $g(z)$ is at least one.

The infinite order of $F(z)$ follows from PÓLYA's result just as in Theorem 3.

The example $F = \exp(z + e^z)$, $f = e^z$, $g = z + e^z$, $h = ze^z$ shows that order 1 can indeed occur for both f and g. In this case $p = 1$.

References

[1] I. N. BAKER, Entire functions with linearly distributed values, *Math. Z.*, **86** (1964), 263—267.
[2] L. BIEBERBACH, Zwei Sätze über das Verhalten analytischer Funktionen in der Umgebung wesentlich singulärer Stellen, *Math. Z.*, **2** (1918), 158—170.
[3] R. BOAS, *Entire functions* (New York, 1954).
[4] M. L. CARTWRIGHT, Some inequalities in the theory of functions, *Math. Ann.*, **111** (1935), 98—118.
[5] R. NEVANLINNA, *Eindeutige analytische Funktionen* (Berlin, 1953).
[6] G. PÓLYA, On an integral function of an integral function, *J. London Math. Soc.*, **1** (1926), 12—15.
[7] A. RÉNYI and C. RÉNYI, Some remarks on periodic entire functions, *J. Analyse Math.*, **14**, (1965), 303—310.

(Received March 15, 1966)

A note on operators whose spectrum is a spectral set

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The reader is referred to [6] and [7] for terminology, and for the basic properties of spectral sets. If A is an operator we write $\sigma(A)$ for the spectrum of A, and $r(A)$ for the spectral radius of A. If S is a compact set of complex numbers, and f is an S-analytic function, we write $\|f\|_S = \sup\{|f(\lambda)| : \lambda \in S\}$. Thus, to say that S is a spectral set for A means: (i) $\sigma(A) \subset S$, and (ii) $\|f(A)\| \leqq \|f\|_S$ for every rational function f with no poles in S.

A key result proved in [6] is the following (VON NEUMANN's Spectral Mapping Theorem): *If S is a spectral set for the operator A and f is any S-analytic function, then $f(S)$ is a spectral set for the operator $f(A)$* ([6, p. 226, 3.4 (ii)]; see [2] for a recent exposition). It is implicit in VON NEUMANN's theorem that $\sigma[f(A)] \subset f(S)$. Moreover, C. FOIAŞ has shown that the "spectral mapping formula" holds:

$$\sigma[f(A)] = f[\sigma(A)], \tag{1}$$

where f is any S-analytic function, S being a spectral set for A [4, p. 369, (i)]. The first aim of this note is to present an elementary proof of (1) in the special case that $S = \sigma(A)$ (of course this places a restriction on the operator A):

Theorem 1. *If $\sigma(A)$ is a spectral set for the operator A, then* (1) *holds for every $\sigma(A)$-analytic function f.*

The proof is based on a general lemma:

Lemma. *If S is a spectral set for the operator A, then*

$$f[\sigma(A)] \subset \sigma[f(A)] \subset f(S) \tag{2}$$

for every S-analytic function f.

Proof. Assuming $\lambda \in \sigma(A)$, let us show $f(\lambda) \in \sigma[f(A)]$. Let f_n be a sequence of rational functions with no poles in S, such that $f_n \to f$ uniformly on S. Then $f_n(A) \to f(A)$ in norm [6, p. 264, 3.3. (I)]; since also $f_n(\lambda) \to f(\lambda)$, we have

$$f_n(A) - f_n(\lambda)I \to f(A) - f(\lambda)I \quad \text{in norm.}$$

By the spectral mapping formula for rational functions (which is no deeper than the spectral mapping formula for polynomial functions), we have

$$f_n(\lambda) \in f_n[\sigma(A)] = \sigma[f_n(A)];$$

thus the operators $f_n(A) - f_n(\lambda)I$ are singular, hence $f(A) - f(\lambda)I$ is also singular.

Proof of Theorem 1. Put $S=\sigma(A)$ in (2).

Corollary 1. *If $\sigma(A)$ is a spectral set for the operator A and f is any $\sigma(A)$-analytic function, then $\sigma[f(A)]$ is a spectral set for $f(A)$.*

Proof. By von Neumann's theorem, $f[\sigma(A)]$ is a spectral set for $f(A)$; cite formula (1).

An operator A is called *normaloid* if

$$\|A\|=\sup\{|Ax, x)|: \|x\|=1\};$$

this is equivalent to the condition $r(A)=\|A\|$ by an elementary argument [3, proof of Theorem 3].

Corollary 2. *If $\sigma(A)$ is a spectral set for the operator A and f is any $\sigma(A)$-analytic function, then $f(A)$ is normaloid.*

Proof. Since the function $u(\lambda)\equiv\lambda$ is $\sigma(A)$-analytic, we have $\|A\|=\|u(A)\|\leqq \leqq\|u\|_{\sigma(A)}=r(A)$; but $r(A)\leqq\|A\|$ (for any operator), thus $r(A)=\|A\|$, and so A is normaloid. By Corollary 1, the same argument is applicable to $f(A)$, thus $f(A)$ is normaloid.

The special case of Corollary 2 for rational functions f with no poles in $\sigma(A)$ was proved by S. Hildebrandt [5, p. 421, Corollary]. The following result, related to Corollary 2, is much more elementary; it is implicit in [5, p. 420, Remark]

Theorem 2. *In order that $\sigma(A)$ be a spectral set for the operator A, it is necessary and sufficient that $f(A)$ be normaloid for every rational function f with no poles in $\sigma(A)$.*

Proof. If f is a rational function with no poles in $\sigma(A)$, then $\sigma[f(A)]=f[\sigma(A)]$ by elementary considerations, and so $r[f(A)]=\|f\|_{\sigma(A)}$.

Suppose first that $\sigma(A)$ is a spectral set for A. If f is any rational function with no poles in $\sigma(A)$, then $\|f(A)\|\leqq\|f\|_{\sigma(A)}=r[f(A)]\leqq\|f(A)\|$, thus $f(A)$ is normaloid.

Conversely, if $f(A)$ is normaloid for all rational functions f with no poles in $\sigma(A)$, then $\|f(A)\|=r[f(A)]=\|f\|_{\sigma(A)}$ for all such f, thus $\sigma(A)$ is a spectral set for A

An operator A is called *hyponormal* if $AA^*\leqq A^*A$.

Corollary 1. *If A is an operator such that $f(A)$ is hyponormal for all rational functions f with no poles in $\sigma(A)$, then $\sigma(A)$ is a spectral set for A.*

Proof. It suffices to cite the theorem, due to T. Andô [1], that a hyponormal operator is normaloid. Incidentally, here is an elementary proof of Andô's theorem that avoids any reference to spectrum: if A is hyponormal, then $\|A^n\|=\|A\|^n$ for all positive integers n [8, proof of Theorem 1], and so A is normaloid [3, proof of Theorem 2].

Corollary 2. (von Neumann) *If A is a normal operator, then $\sigma(A)$ is a spectral set for A.*

Proof. Since $f(A)$ is obviously normal for every rational function f with no poles in $\sigma(A)$, the assertion is immediate from Corollary 1. We remark that the proof does not use the spectral theorem (*cf.* [6, p. 277]).

I am grateful to Professor Sz.-Nagy for calling my attention to the reference [4].

References

[1] T. Andô, On hyponormal operators, *Proc. Amer. Math. Soc.*, **14** (1963), 290—291.

[2] H. Baumgärtel—S. K. Berberian, Bemerkung zu einem Satz von J. v. Neumann. To appear in *Math. Nachr.*

[3] S. J. Bernau—F. Smithies, A note on normal operators, *Proc. Cambridge Philos. Soc.*, **59** (1963), 727—729.

[4] C. Foiaş, Unele aplicaţii ale mulţimilor spectrale. I. Măsura armonică-spectrală, *Studii şi cercetări matematice,* **10** (1959), 365—401.

[5] S. Hildebrandt, The closure of the numerical range of an operator as spectral set, *Comm. Pure Appl. Math.*, **17** (1964), 415—421.

[6] J. v. Neumann, Eine Spektraltheorie für allgemeine Operatoren eines unitären Raumes, *Math. Nachr.*, **4** (1950—51), 258—281.

[7] F. Riesz—B. Sz.-Nagy, *Leçons d'analyse fonctionnelle* (Budapest, 1952).

[8] J. G. Stampfli, Hyponormal operators, *Pacific J. Math.*, **12** (1962), 1453—1458.

(Received December 10, 1965)

Über verschiedene Konvergenzarten trigonometrischer Reihen. III
(Bedingungen in der Metrik von L^p)

Von LÁSZLÓ LEINDLER in Szeged

Einleitung

Es sei $f(x) \in L^p(0, 2\pi)$ $(1 < p < \infty)$ eine 2π-periodische Funktion mit der Fourier-Entwicklung

(1) $$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv S[f].$$

$E_n(p) = E_n(f, p)$ bezeichne den besten Annäherungsgrad von $f(x)$ im Sinne der Metrik von $L^p(0, 2\pi)$ mit trigonometrischen Polynomen $(n-1)$-ter Ordnung.

Wir setzen

$$\Delta_s^k f(x, h) = \sum (-1)^{k-j} C_k^j f\left(x - (k-2j)\frac{h}{2}\right)$$

und

$$\omega_k^{(p)}(\delta) = \omega_k^{(p)}(f, \delta) = \sup_{0 \leqq t \leqq \delta} \left\{\int_0^{2\pi} |\Delta_s^k f(x, t)|^p \, dx\right\}^{1/p},$$

ferner bezeichnen wir mit $\Lambda(\alpha, \gamma_1, \gamma_2)$ die Klasse der monotonen Funktionen $\lambda(x)$ $(x \geqq 1)$, für die $x^\alpha(\log(x+1))^{\gamma_1} \leqq \lambda(x) \leqq x^\alpha(\log(x+1))^{\gamma_2}$.

In zwei früheren Arbeiten ([1], [2]) haben wir u.a. verschiedene hinreichende Strukturbedingungen für verschiedene Arten der Konvergenz von (1) in der Metrik von $L^2(0, 2\pi)$ angegeben. Im vorliegenden Aufsatz werden wir den allgemeineren Fall von $L^p(0, 2\pi)$ betrachten $(1 < p < \infty)$.

Wenn wir im Folgenden über die Existenz der r-ten Ableitung einer Funktion $f(x)$ sprechen (r eine natürliche Zahl), so verstehen wir, daß $f(x)$ fast überall gleich der r-fach iterierten Integralfunktion einer quadratisch integrierbaren Funktion $g(x)$ ist, und dann nennen wir $g(x)$ die r-te Ableitung von $f(x)$, in Formel: $g(x) = f^{(r)}(x)$.

Wir setzen zur Abkürzung:

$$A_k(t) = A_k(f, p; t) = \left(\int_0^{2\pi} |\Delta_s^k f(x, t)|^p \, dx\right)^{1/p}.$$

Satz I. *Sei* $1<p\leqq 2$ *und seien* r *und* k *nichtnegative ganze Zahlen mit* $k>r$.

α) *Unter der Bedingung*

$$\int_0^1 \frac{|\log t|\, A_k(t)}{t^{\frac{1}{2}+\frac{1}{p}+r}}\, dt<\infty \tag{2}$$

existiert $f^{(r)}(x)$ *und konvergiert ihre Fourier-Entwicklung fast überall unbedingt, d.h. bei jeder Anordnung ihrer Glieder.*

β) *Unter der Bedingung*

$$\int_0^1 \frac{|\log t|^{1/2} A_k(t)}{t^{\frac{1}{2}+\frac{1}{p}+r}}\, dt<\infty \tag{3}$$

existiert $f^{(r)}(x)$ *und konvergieren die Reihen*

$$\frac{A_0}{2}+\sum_{n=1}^{\infty} \pm (A_n \cos nx + B_n \sin nx) \quad \left(\begin{matrix} A_n \\ B_n \end{matrix} = \frac{1}{\pi}\int_0^{2\pi} f^{(r)}(x) \begin{matrix} \cos nx \\ \sin nx \end{matrix}\, dx\right)$$

bei fast allen Vorzeichenverteilungen gleichmäßig.

γ) *Unter jeder Bedingung*

$$\int_0^1 \frac{A_k(t)}{t^{\frac{1}{2}+\frac{1}{p}+r}}\, dt<\infty, \tag{4}$$

$$\int_0^1 \frac{|\log t|^{1/2} A_k(t)}{t^{\frac{1}{2}+\frac{1}{p}+r}}\, dt<\infty, \tag{5}$$

$$\int_0^1 \frac{A_k(t)}{t^{1+\frac{1}{p}+r-\alpha}}\, dt<\infty \tag{6}$$

existiert $f^{(r)}(x)$ *und ist ihre Fourier-Entwicklung fast überall* $|C, \alpha>\frac{1}{2}|$-, $|C, \frac{1}{2}|$-, *bzw.* $|C, \alpha|$-*summierbar*)* $(-1<\alpha<\frac{1}{2})$.

δ) *Unter der Bedingung*

$$\int_0^1 \frac{|\log t|\, A_k^2(t)}{t^{2r+\frac{2}{p}}}\, dt<\infty \tag{7}$$

existiert $f^{(r)}(x)$ *und konvergiert ihre Fourier-Entwicklung fast überall.*

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*) Eine Reihe Σu_n heißt $|C, \alpha|$-summierbar, wenn

$$\sum_{n=1}^{\infty} |\sigma_{n+1}^{(\alpha)}-\sigma_n^{(\alpha)}|<\infty$$

gilt, wobei $\sigma_n^{(\alpha)}$ das n-te (C, α)-Mittel bezeichnet.

Zum Beweis des Satzes I werden wir aus den angegebenen Strukturbedingungen gewisse Bedingungen von der Form

$$\sum_{n=1}^{\infty} \frac{1}{\lambda(n)} R_n^\beta(p') < \infty \qquad \left(R_n(p') = \left\{ \sum_{k=n}^{\infty} (a_k^2 + b_k^2)^{p'/2} \right\}^{1/p'}, \quad p' = \frac{p}{p-1} \right)$$

herleiten, woraus durch Anwendung bekannter Koeffizientenbedingungen die entsprechenden Behauptungen folgen.

Mit Rücksicht auf das Ergebnis von A. F. TIMAN und M. F. TIMAN [13]:

$$R_n(p') \leqq K_1 E_n(p) \qquad (\text{für } 1 < p \leqq 2),$$

bzw. das Ergebnis von STEČKIN [11]:

$$E_n(p) \leqq K_2\, \omega_2^{(p)}\left(\frac{1}{n}\right),$$

bekommen wir aus dem Satz I den folgenden:

Satz II. *Sei* $1 < p \leqq 2$. *Die entsprechenden Behauptungen des Satzes* I *bleiben mit*

$$(2') \quad \sum_{n=1}^{\infty} (\log n)\, n^{-\frac{3}{2}+\frac{1}{p}+r} E_n(p) < \infty \quad \text{oder} \quad \sum_{n=1}^{\infty} (\log n)\, n^{-\frac{3}{2}+\frac{1}{p}+r} \omega_2^{(p)}\left(\frac{1}{n}\right) < \infty;$$

$$(3') \quad \sum_{n=1}^{\infty} \sqrt{\log n}\, n^{-\frac{3}{2}+\frac{1}{p}+r} E_n(p) < \infty \quad \text{oder} \quad \sum_{n=1}^{\infty} \sqrt{\log n}\, n^{-\frac{3}{2}+\frac{1}{p}+r} \omega_2^{(p)}\left(\frac{1}{n}\right) < \infty;$$

$$(4') \quad \sum_{n=1}^{\infty} n^{-\frac{3}{2}+\frac{1}{p}+r} E_n(p) < \infty \quad \text{oder} \quad \sum_{n=1}^{\infty} n^{-\frac{3}{2}+\frac{1}{p}+r} \omega_2^{(p)}\left(\frac{1}{n}\right) < \infty;$$

$$(5') \quad \sum_{n=1}^{\infty} \sqrt{\log n}\, n^{-\frac{3}{2}+\frac{1}{p}+r} E_n(p) < \infty \quad \text{oder} \quad \sum_{n=1}^{\infty} \sqrt{\log n}\, n^{-\frac{3}{2}+\frac{1}{p}+r} \omega_2^{(p)}\left(\frac{1}{n}\right) < \infty;$$

$$(6') \quad \sum_{n=1}^{\infty} n^{-1+\frac{1}{p}+r-\alpha} E_n(p) < \infty \quad \text{oder} \quad \sum_{n=1}^{\infty} n^{-1+\frac{1}{p}+r-\alpha} \omega_2^{(p)}\left(\frac{1}{n}\right) < \infty;$$

$$(7') \quad \sum_{n=1}^{\infty} (\log n)\, n^{-2+\frac{2}{p}+2r} E_n^2(p) < \infty \quad \text{oder} \quad \sum_{n=1}^{\infty} (\log n)\, n^{-2+\frac{2}{p}+2r} \left(\omega_2^{(p)}\left(\frac{1}{n}\right)\right)^2 < \infty$$

anstatt (2), ..., (7) *gültig*.

Bekanntlich hat MARCINKIEWICZ [6] den Satz von PLESSNER [8] folgenderweise verallgemeinert: Unter der Bedingung

$$\int_0^{2\pi}\int_0^{2\pi} \frac{|f(x+t)-f(x-t)|^p}{t}\, dx\, dt < \infty,$$

für ein p mit $1 < p \leqq 2$, konvergiert die Entwicklung (1) fast überall

POTAPOV [9] hat neulich bewiesen, daß für $p>1$ die drei Bedingungen

$$\left\{\begin{array}{l} \displaystyle\int_0^{2\pi}\int_0^{2\pi} \frac{|f(x+t)-f(x-t)|^p}{t}\,dx\,dt<\infty, \\ \displaystyle\sum_{n=1}^{\infty}\frac{1}{n}E_n^p(p)<\infty, \quad \int_0^{2\pi}\frac{\omega_1^{(p)}(f,t)^p}{t}\,dt<\infty \end{array}\right. \tag{8}$$

paarweise äquivalent sind. (Er hat aber diese Behauptung in dieser Form nicht ausgesprochen.)

Unlängst haben wir [3] einen Äquivalenzsatz bewiesen, der in einem Spezialfall folgenderweise lautet:

Sei $0<\beta\leqq 2$ und sei $\lambda(x)\in\Lambda(\alpha,\gamma_1,\gamma_2)$ mit $\alpha>1-\beta$ und mit gewissen $\gamma_1<\gamma_2$. Dann sind die drei Bedingungen

$$\int_0^1 \frac{1}{t^2\lambda\left(\frac{1}{t}\right)}\left(\int_0^{2\pi}[f(x+2t)+f(x-2t)-2f(x)]^2\,dx\right)^{\beta/2}dt<\infty,$$

$$\sum_{n=1}^{\infty}\frac{1}{\lambda(n)}E_n^{\beta}(2)<\infty, \qquad \sum_{n=1}^{\infty}\frac{1}{\lambda(n)}\omega_2^{(2)}\left(\frac{1}{n}\right)^{\beta}<\infty$$

paarweise äquivalent.

Der Vergleich dieser Ergebnisse stellt die Frage, ob ein Äquivalenzsatz von obiger Art im Raum $L^p(0,2\pi)$ mit $1<p<\infty$ gilt. Der folgende Satz gibt für diese Frage eine positive Antwort.

Satz III. *Seien $p>1$ und $\beta\geqq 1$ reelle Zahlen, weiterhin sei $\lambda(x)\in\Lambda(\alpha,\gamma_1,\gamma_2,)$ mit $\alpha>\max(1-\beta,-2)$ und mit gewissen $\gamma_1<\gamma_2$. Dann sind die drei Bedingungen*

$$\int_0^1 \frac{1}{t^2\lambda\left(\frac{1}{t}\right)}\left(\int_0^{2\pi}|f(x+2t)+f(x-2t)-2f(x)|^p\,dx\right)^{\beta/p}dt<\infty, \tag{9}$$

$$\sum_{n=1}^{\infty}\frac{1}{\lambda(n)}E_n^{\beta}(f,p)<\infty, \tag{10}$$

$$\sum_{n=1}^{\infty}\frac{1}{\lambda(n)}\omega_2^{(p)}\left(f,\frac{1}{n}\right)^{\beta}<\infty \tag{11}$$

paarweise äquivalent.

Es ist bekannt (s. z. B. [12], s. 339), daß für jedes p mit $1<p<\infty$ gilt:

$$E_n(p)\sim\|f(x)-S_{n-1}(x)\|_p.$$

So ist es ersichtlich, daß man auf Grund des Satzes von HAUSDORFF—YOUNG, ([14], II, s. 101.) bzw. von HARDY—LITTLEWOOD ([14], II, s. 128.) auch notwendige und hinreichende, durch Koeffizienten angegebene Bedingungen für das Erfülltsein der Bedingungen (9), (10) und (11) angeben kann. Im Falle $1<p\leqq 2$ sind die Bedingungen

$$\sum_{n=1}^{\infty}\frac{1}{\lambda(n)}\left\{\sum_{k=n}^{\infty}(|a_k|^{p'}+|b_k|^{p'})\right\}^{\beta/p'}<\infty \tag{12}$$

und

$$\sum_{n=1}^{\infty}\frac{1}{\lambda(n)}\left\{\sum_{k=n}^{\infty}(|a_k|^{p}+|b_k|^{p})k^{p-2}\right\}^{\beta/p}<\infty . \tag{13}$$

notwendig, und im Falle $2\leqq p<\infty$ sind die obigen Bedingungen hinreichend dafür, daß die Bedingungen (9)—(11) erfüllt sind. Wir bemerken noch, daß die Bedingungen (12) und (13) im allgemeinen unvergleichbar sind, wenn aber die Folge $\varrho_n=(a_n^2+b_n^2)^{\frac{1}{2}}$ monoton ist, folgt für $1<p\leqq 2$ (12) aus (13) und für $p\geqq 2$ folgt (13) aus (12).

§ 1. Hilfssätze

Hilfssatz I. *Ist $f(x)\in L^p(0,2\pi)$, dann gilt*

$$\omega_1^{(p)}\left(f,\frac{1}{n}\right)\leqq\frac{C}{n}\sum_{\nu=0}^{n}E_\nu(f,p).$$

Dieser Hilfssatz ist bekannt. (Siehe z.B. [12], S. 344.)

Hilfssatz II. *Seien $\lambda(x)$ $(x\geqq 1)$ eine positive, monotone Funktion mit $\lambda(n)\leqq \leqq A\lambda(2n)$ $(A\geqq 2,\ n=1,2,\ldots)$ und p, β reelle Zahlen mit*

$$1<p\leqq 2 \quad \textit{und} \quad 0<\beta\leqq p'=\frac{p}{p-1}.$$

Dann folgt die Ungleichung

$$\sum_{n=1}^{\infty}\frac{1}{\lambda(n)}\left\{\sum_{k=n}^{\infty}\varrho_k^{p'}\right\}^{\beta/p'}<\infty \qquad (\varrho_k^2=a_k^2+b_k^2)$$

aus

$$\int_0^1\frac{1}{t^2\lambda\left(\frac{1}{t}\right)}\left(\int_0^{2\pi}|\Delta_s^k f(x,t)|^p\,dx\right)^{\beta/p}dt<\infty \qquad (k\geqq 1).$$

Dieser Hilfssatz kann analog zu dem Beweis des Hilfssatzes III von [3] bewiesen werden, nur soll man die Beziehung

$$\Delta_s^k f(x,t)=2^k\sum_{n=1}^{\infty}\left\{a_n\cos\left(nx+k\frac{\pi}{2}\right)+b_n\sin\left(nx+k\frac{\pi}{2}\right)\right\}(\sin nt)^k$$

anstatt der Beziehung bezüglich $\Delta_s^2 f(x,2t)$ benutzen.

Hilfssatz III. *Seien $\{a_n\}$ und $\{\alpha_n\}$ nicht-negative Zahlenfolgen mit*

$$\sum_{k=n}^{\infty} \alpha_k = \beta_n \alpha_n.$$

Dann gilt für jedes $p \geqq 1$

$$\sum_{k=1}^{\infty} \alpha_k \left(\sum_{n=1}^{k} a_n\right)^p \leqq p^p \sum_{k=1}^{\infty} \alpha_k (\beta_k a_k)^p. \tag{1.1}$$

Beweis. Mit der Abkürzung $s_k = \sum_{n=1}^{k} a_n$ $(s_0 = 0)$ gilt für jedes N

$$\sum_{k=1}^{N} \alpha_k s_k^p = \sum_{k=1}^{N} (\beta_k \alpha_k - \beta_{k+1} \alpha_{k+1}) s_k^p \leqq$$

$$\leqq \sum_{k=1}^{N} \beta_k \alpha_k (s_k^p - s_{k-1}^p) \leqq p \sum_{k=1}^{N} \beta_k \alpha_k s_k^{p-1} a_k \leqq$$

$$\leqq p \sum_{k=1}^{N} \alpha_k^{\frac{1}{p}} \beta_k a_k \alpha_k^{1-\frac{1}{p}} s_k^{p-1} \leqq p \left\{\sum_{k=1}^{N} \alpha_k (\beta_k a_k)^p\right\}^{1/p} \left\{\sum_{k=1}^{N} \alpha_k s_k^p\right\}^{\frac{p-1}{p}}.$$

Daraus ergibt sich

$$\left\{\sum_{k=1}^{N} \alpha_k s_k^p\right\}^{1/p} \leqq p \left\{\sum_{k=1}^{N} \alpha_k (\beta_k a_k)^p\right\}^{1/p},$$

woraus die Behauptung (1. 1) unmittelbar folgt.

§ 2. Beweis der Sätze

Beweis von Satz I. α) Auf Grund des Hilfssatzes II, mit $\lambda(x) = x^{\frac{3}{2} - \frac{1}{p} - r} \cdot \log^{-1}(x+1)$, ergibt sich aus der Bedingung (2)

$$\sum_{n=2}^{\infty} n^{-\frac{3}{2} + \frac{1}{p} + r} (\log n) R_n(p') < \infty \qquad \left(p' = \frac{p}{p-1}\right).$$

Daraus erhalten wir:

$$\sum_{n=3}^{\infty} \frac{1}{n} \left\{\sum_{k=n}^{\infty} k^{2r} \varrho_k^2\right\}^{1/2} \leqq 2 \sum_{m=1}^{\infty} \left\{\sum_{k=2^m+1}^{\infty} k^{2r} \varrho_k^2\right\}^{1/2} \leqq$$

$$\leqq 2^{r+1} \sum_{m=1}^{\infty} \sum_{\nu=m}^{\infty} 2^{\nu r} \left\{\sum_{k=2^\nu+1}^{2^{\nu+1}} \varrho_k^2\right\}^{1/2} \leqq 2^{r+1} \sum_{m=1}^{\infty} \sum_{\nu=m}^{\infty} 2^{\nu r} \left\{\sum_{k=2^\nu+1}^{2^{\nu+1}} \varrho_k^{p'}\right\}^{1/p'} 2^{\nu \frac{p'-2}{2p'}} \leqq$$

$$\leqq 2^{r+1} \sum_{\nu=1}^{\infty} \nu 2^{\nu\left(r + \frac{1}{2} - \frac{1}{p'}\right)} R_{2^\nu}(p') \leqq 2^{r+1} \sum_{\nu=1}^{\infty} \nu 2^{\nu\left(r - \frac{3}{2} + \frac{1}{p}\right)} 2^\nu R_{2^\nu}(p') \leqq \tag{2.1}$$

$$\leqq 2^{2r+3} \sum_{n=2}^{\infty} n^{-\frac{3}{2} + \frac{1}{p} + r} (\log n) R_n(p') < \infty.$$

Dies ergibt insbesondere

$$\sum_{k=1}^{\infty} \varrho_k^2 k^{2r} < \infty,$$

folglich existiert $f^{(r)}(x)$ und gehört zu $L^2(0, 2\pi)$. Somit gilt nach einem bekannten Satz (s. z. B. [14], S. 40): $S^{(r)}[f] = S[f^{(r)}]$,*) also

$$E_n(f^{(r)}, 2) = \left\{\sum_{k=n}^{\infty} k^{2r} \varrho_k^2\right\}^{1/2}. \tag{2.2}$$

Daraus und aus (2. 1) folgt die Ungleichung

$$\sum_{n=1}^{\infty} \frac{1}{n} E_n(f^{(r)}, 2) < \infty,$$

welche nach einem Satz des Verfassers [4] die unbedingte Konvergenz von $S[f^{(r)}]$ fast überall nach sich zieht.

Im Falle der Behauptung β) genügt es, nach einem Satz von SALEM und ZYGMUND [10], zu zeigen, daß

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^{1/2}} E_n(f^{(r)}, 2) < \infty. \tag{2.3}$$

Nach dem Hilfssatz II, mit $\lambda(x) = x^{\frac{3}{2}-\frac{1}{p}-r} (\log(x+1))^{-\frac{1}{2}}$, folgt aus (3):

$$\sum_{n=2}^{\infty} n^{-\frac{3}{2}+\frac{1}{p}+r} (\log n)^{1/2} R_n(p') < \infty.$$

Daraus kann man durch eine einfache, zu (2. 1) analoge Rechnung zeigen, daß $f^{(r)}(x)$ existiert und zu $L^2(0, 2\pi)$ gehört; somit gelten $S^{(r)}[f] = S[f^{(r)}]$ und (2. 2), weiterhin auch die Ungleichung (2. 3).

γ) Aus (4), durch Anwendung des Hilfssatzes II mit $\lambda(x) = x^{\frac{3}{2}-\frac{1}{p}-r}$, folgt

$$\sum_{n=1}^{\infty} n^{-\frac{3}{2}+\frac{1}{p}+r} R_n(p') < \infty.$$

Hieraus ergibt sich

$$\sum_{m=1}^{\infty} \left(\sum_{k=2^m+1}^{2^{m+1}} \varrho_k^2 k^{2r}\right)^{1/2} < \infty \tag{2.4}$$

durch die folgende Abschätzung:

$$\sum_{m=1}^{\infty} \left(\sum_{k=2^m+1}^{2^{m+1}} \varrho_k^2 k^{2r}\right)^{1/2} \leqq 2 \sum_{m=1}^{\infty} 2^{mr} \left(\sum_{k=2^m+1}^{2^{m+1}} \varrho_k^{p'}\right)^{1/p'} 2^{m\left(\frac{1}{2}-\frac{1}{p'}\right)} \leqq$$

$$\leqq 2 \sum_{m=1}^{\infty} 2^{m\left(r+\frac{1}{p}-\frac{3}{2}\right)} 2^m R_{2^m}(p') \leqq 2^{r+2} \sum_{n=1}^{\infty} n^{-\frac{3}{2}+\frac{1}{p}+r} R_n(p').$$

*) $S^{(r)}[f]$ bezeichnet die aus der Fourierreihe $S[f]$ durch r-malige gliedweise Ableitung entstandene Reihe.

Aus (2. 4) folgt, daß $f^{(r)}(x)\in L^2(0, 2\pi)$ und so ist $S^{(r)}[f]=S[f^{(r)}]$. Mit Rücksicht auf die letzte Behauptung zieht die Ungleichung (2. 4) nach einem Satz von [5] die $|C, \alpha>\frac{1}{2}|$-Summierbarkeit von $S[f^{(r)}]$ fast überall nach sich.

Ähnlicherweise können die beiden anderen Behauptungen eingesehen werden. Nämlich folgt aus (5) bzw. aus (6)

$$\sum_{n=1}^{\infty} n^{-\frac{3}{2}+\frac{1}{p}+r}(\log n)^{1/2} R_n(p')<\infty$$

bzw.

$$\sum_{n=1}^{\infty} n^{-1+\frac{1}{p}+r-\alpha} R_n(p')<\infty,$$

nach dem Hilfssatz II, woraus man durch einfache Rechnung erhält:

$$\sum_{m=1}^{\infty} \sqrt{m}\left(\sum_{k=2^m+1}^{2^{m+1}} \varrho_k^2 k^{2r}\right)^{1/2}<\infty$$

und

$$\sum_{m=1}^{\infty} 2^{\frac{m}{2}(1-2\alpha)}\left(\sum_{k=2^m+1}^{2^{m+1}} \varrho_k^2 k^{2r}\right)^{1/2}<\infty.$$

Hieraus folgt, daß $S[f^{(r)}]$ fast überall $|C, \frac{1}{2}|$- bzw. $|C, -1<\alpha<\frac{1}{2}|$-summierbar ist (s. [5], Satz II).

δ) Aus (7), auf Grund des Hilfssatzes II mit $\lambda(x)=x^{2-\frac{1}{p}+2r}(\log(x+1))^{-1}$ und $\beta=2$, ergibt sich

$$\sum_{n=2}^{\infty} n^{-2+\frac{2}{p}+2r}(\log n) R_n^2(p')<\infty. \tag{2. 5}$$

Hieraus folgt, daß

$$\sum_{k=1}^{\infty} \varrho_k^2 k^{2r}\log k<\infty; \tag{2. 6}$$

dies sichert, nach dem wohlbekannten Kolmogoroff—Seliverstoff—Plessnerschen Satz die Konvergenz von $S[f^{(r)}]$ fast überall; es gilt nämlich $S^{(r)}[f]=S[f^{(r)}]$ auch in diesem Fall. Die Implikation (2. 5)$\Rightarrow$(2. 6) können wir folgenderweise beweisen:

$$\sum_{k=3}^{\infty} \varrho_k^2 k^{2r}\log k \leqq 2^3 \sum_{n=3}^{\infty}\frac{1}{n}\sum_{k=n}^{\infty}\varrho_k^2 k^{2r} \leqq 2^{2r+3}\sum_{m=1}^{\infty}\sum_{n=2^m+1}^{2^{m+1}}\frac{1}{n}\sum_{\nu=m}^{\infty}2^{2\nu r}\sum_{k=2^\nu+1}^{2^{\nu+1}}\varrho_k^2 \leqq$$

$$\leqq 2^{2r+3}\sum_{m=1}^{\infty} m 2^{m\left(2r+1-\frac{2}{p'}\right)}\left\{\sum_{k=2^m+1}^{2^{m+1}}\varrho_k^{p'}\right\}^{2/p'} \leqq 2^{4r+6}\sum_{n=1}^{\infty} n^{2r+\frac{2}{p}-2}(\log n) R_n^2(p').$$

Damit haben wir den Satz I vollständig bewiesen.

Beweis von Satz III.*) Zuerst beweisen wir die Implikation (9)$\Rightarrow$(10). Im Falle $\sum \lambda^{-1}(n) < \infty$ ist die Behauptung trivial. Wir nehmen also an, daß $\alpha \leqq 1$. Sei

$$u_n(x) = \frac{3}{2\pi n(2n^2+1)} \int_0^{2\pi} f(t) \left(\frac{\sin \frac{t-x}{2} n}{\sin \frac{t-x}{2}} \right)^4 dt =$$

$$= \frac{3}{\pi n(2n^2+1)} \int_0^{\pi/2} [f(x+2t)+f(x-2t)] \left(\frac{\sin nt}{\sin t} \right)^4 dt.$$

$u_n(x)$ ist ein trigonometrisches Polynom $(2n-2)$-ten Grades und man hat

$$\int_0^{\pi/2} \frac{6}{\pi n(2n^2+1)} \left(\frac{\sin nt}{\sin t} \right)^4 dt = 1$$

(s. [7], S, 113—116). Nach dem Obigen ist es klar, daß $E_{2^{k+1}}(f, p) \leqq \|f(x) - u_{2^k}(x)\|_p$, d.h. gilt

$$E_{2^{k+1}}(f, p) \leqq \left\{ \int_0^{2\pi} \left| \int_0^{\pi/2} [f(x+2t) - 2f(x) + f(x+2t)] \frac{3}{\pi 2^k (2^{2k+1}+1)} \cdot \right.\right.$$

$$\left.\left. \cdot \left(\frac{\sin 2^k t)}{\sin t} \right)^4 dt \right|^p dx \right\}^{1/p}.$$

Durch Anwendung der Minkowskischen Ungleichung bekommt man

$$E_{2^{k+1}}(f, p) \leqq 2^{-3k} \int_0^{\pi/2} \left(\frac{\sin 2^k t}{\sin t} \right)^4 \left\{ \int_0^{2\pi} |f(x+2t) - 2f(x) + f(x-2t)|^p \, dx \right\}^{1/p} dt \equiv$$

$$\equiv 2^{-3k} \int_0^{\pi/2} \left(\frac{\sin 2^k t}{\sin t} \right)^4 A(t) \, dt.$$

Für $\beta > 1$ erhalten wir durch Anwendung der Hölderschen Ungleichung

$$E^{\beta}_{2^{k+1}}(f, p) \leqq K_1 \left\{ \int_0^{\pi/2} \left[2^{-3k} \left(\frac{\sin 2^k t}{\sin t} \right)^4 A(t)^{\beta} \right]^{1/\beta} \cdot \right.$$

$$\left. \cdot \left[\frac{6}{\pi 2^k (2^{2k+1}+1)} \left(\frac{\sin 2^k t}{\sin t} \right)^4 \right]^{1-\frac{1}{\beta}} dt \right\}^{\beta} \leqq K_1 \int_0^{\pi/2} 2^{-3k} \left(\frac{\sin 2^k t}{\sin t} \right)^4 A(t)^{\beta} \, dt.$$

*) Unser Beweis ist ähnlich zum Lemma 6 von POTAPOV [9].

Daraus folgt für jedes $\beta\geqq 1$

$$\sum_{n=2}^{\infty}\frac{1}{\lambda(n)}E_n^{\beta}(f,p)\leqq K_2\sum_{m=0}^{\infty}\frac{2^m}{\lambda(2^m)}E_{2^{m+1}}^{\beta}(f,p)\leqq$$

$$\leqq K_3\int_0^{\pi/2}\frac{A(t)^{\beta}}{t^2\lambda\left(\frac{1}{t}\right)}\sum_{m=0}^{\infty}\frac{t^2\lambda\left(\frac{1}{t}\right)}{2^{2m}\lambda(2^m)}\left(\frac{\sin 2^m t}{\sin t}\right)^4 dt.$$

Auf Grund dieser Ungleichungen genügt es für den Beweis der Implikation (9)⇒(10) zu zeigen, daß die Summe

$$\sigma(t)=\sum_{m=0}^{\infty}\frac{t^2\lambda\left(\frac{1}{t}\right)}{2^{2m}\lambda(2^m)}\left(\frac{\sin 2^m t}{\sin t}\right)^4$$

für $0<t\leqq\frac{\pi}{2}$ gleichmäßig beschränkt ist. Für ein beliebiges, aber fixiertes t bezeichnen wir mit m_0 die größte unter den natürlichen Zahlen m, mit $2^m t\leqq 1$. Dann ist

$$\sigma(t)=\sum_{m=0}^{m_0}+\sum_{m=m_0+1}^{\infty}\equiv\sigma_1(t)+\sigma_2(t).$$

Es ist leicht ersichtlich, daß $\sigma_1(t)$ beschränkt ist. Wegen $\alpha\leqq 1$ ist nämlich

$$\sigma_1(t)\leqq K_4\sum_{m=0}^{m_0}\frac{\lambda\left(\frac{1}{t}\right)}{\lambda(2^m)}2^{2m}t^2=K_4t^2\lambda\left(\frac{1}{t}\right)\sum_{m=0}^{m_0}\frac{2^{2m}}{\lambda(2^m)}\leqq$$

$$\leqq K_5t^2\lambda\left(\frac{1}{t}\right)\frac{2^{2m_0}}{\lambda(2^{m_0})}\leqq K_6.$$

Wegen $\alpha>-2$ ist ferner

$$\sigma_2(t)\leqq K_7\sum_{m=m_0+1}^{\infty}t^{-2}\lambda\left(\frac{1}{t}\right)2^{-2m}\lambda^{-1}(2^m)\leqq$$

$$\leqq K_8t^{-2}\lambda\left(\frac{1}{t}\right)2^{-2m_0}\lambda^{-1}(2^{m_0})\leqq K_9.$$

Damit haben wir die Implikation (9)⇒(10) bewiesen.

Wir beweisen nun die Implikation (10)⇒(11). Da wegen $\alpha>1-\beta$ und $\lambda(x)\in\Lambda(\alpha,\gamma_1,\gamma_2)$ gilt:

$$\sum_{k=n}^{\infty}\frac{1}{\lambda(k)k^{\beta}}\leqq K_{10}\frac{n}{\lambda(n)n^{\beta}},$$

es ergibt sich aus den Hilfssätzen I und III

$$\sum_{n=1}^{\infty} \frac{1}{\lambda(n)} \omega^{(p)} \left(f, \frac{1}{n}\right)^{\beta} \leqq K_{11} \sum_{n=1}^{\infty} \frac{1}{\lambda(n) n^{\beta}} \left(\sum_{\nu=1}^{n} E_{\nu}(f, p)\right)^{\beta} \leqq$$

$$\leqq K_{12} \sum_{n=1}^{\infty} \frac{1}{\lambda(n) n^{\beta}} (n E_n(f, p))^{\beta},$$

was zu beweisen war.

Was die Implikation (11)$\Rightarrow$(9) anbetrifft, genügt es zu bemerken, daß die Bedingung (11) mit

$$\int_0^1 \frac{1}{t^2 \lambda\left(\frac{1}{t}\right)} \omega_2^{(p)}(f, t)^{\beta}\, dt < \infty \tag{2.7}$$

gleichwertig ist, da (9) aus (2. 7) offenbar folgt.

Damit haben wir den Satz III vollständig bewiesen.

Schriftenverzeichnis

[1] L. Leindler, Über verschiedene Konvergenzarten trigonometrischer Reihen, *Acta Sci. Math.*, **25** (1964), 233—249.
[2] L. Leindler, Über verschiedene Konvergenzarten trigonometrischer Reihen. II, *Acta Sci. Math.*, **26** (1965), 118—124.
[3] L. Leindler, Über Strukturbedingungen für Fourierreihen, *Math. Z.*, **88** (1965), 418—431.
[4] L. Leindler, Über unbedingte Konvergenz der Orthogonalreihen mit strukturellen Bedingungen, *Studia Math.*, **23** (1963), 113—117.
[5] L. Leindler, Über die absolute Summierbarkeit der Orthogonalreihen, *Acta Sci. Math.*, **22** (1961), 243—268.
[6] J. Marcinkiewicz, Sur une nouvelle condition pour la convergence presque partout des séries de Fourier, *Annali della Scuola Normale Superiore di Pisa*, **8** (1939), 239—240.
[7] И. П. Натансон, *Конструктивная теория функций* (Москва—Ленинград, 1949).
[8] A. Plessner, Über Konvergenz von trigonometrischen Reihen, *J. reine angew. Math.*, **155** (1926), 15—25.
[9] M. K. Потапов, К вопросу об эквивалентности условий сходимости рядов Фурье, *Мат. сборник*, **68** (1965), 111—127.
[10] R. Salem—A. Zygmund, Some properties of trigonometric series whose terms have random sings, *Acta Math.*, **91** (1954), 245—301.
[11] С. Б. Стечкин, О порядке наилучших приближений непрерывных функций, *Изв. АН СССР*, **15** (1951), 219—242.
[12] А. Ф. Тиман, *Теория приближения функций действительного переменного* (Москва, 1960).
[13] А. Ф. Тиман—Т. Ф. Тиман, Обобщенный модуль непрерывности и наилучшее приближение в среднем, *Доклады АН СССР*, **71** (1950), 17—20.
[14] A. Zygmund, *Trigonometric series*. I—II (Cambridge, 1959).

(Eingegangen am 11. März 1966)

On the strong summability of orthogonal series

By LÁSZLÓ LEINDLER in Szeged

1. Let $\{\varphi_n(x)\}$ $(n=0, 1, \dots)$ be an orthonormal system on the interval (a, b). We shall consider series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x) \tag{1.1}$$

with real coefficients satisfying

$$\sum_{n=0}^{\infty} c_n^2 < \infty. \tag{1.2}$$

By the Riesz—Fischer theorem, the series (1.1) converges in the mean to a square-integrable function $f(x)$. By $s_n(x)$ and $\sigma_n^\alpha(x)$ we denote the n-th partial sums and the n-th Cesàro means of order $\alpha(>-1)$ of the series (1.1), i.e.

$$s_n(x) = \sum_{\nu=0}^{\infty} c_\nu \varphi_\nu(x)$$

and

$$\sigma_n^\alpha(x) = \frac{1}{A_n^{(\alpha)}} \sum_{\nu=0}^{n} A_{n-\nu}^{(\alpha-1)} s_\nu(x) \qquad \left(A_n^{(\alpha)} = \binom{n+\alpha}{n}\right).$$

2. Concerning the strong and very strong summability of (1.1), SUNOUCHI [3] proved recently the following theorems:

Theorem A. *If the orthogonal series* (1.1) *with* (1.2) *is* $(C, 1)$*-summable to* $f(x)$ *almost everywhere in* (a, b), *then*

$$\lim_{n\to\infty} \frac{1}{A_n^{(\alpha)}} \sum_{\nu=0}^{n} A_{n-\nu}^{(\alpha-1)} |s_\nu(x) - f(x)|^k = 0$$

almost everywhere in (a, b) *for any* $\alpha > 0$ *and* $k > 0$.

Theorem B. *If*

$$\sum_{n=4}^{\infty} c_n^2 (\log\log n)^2 < \infty, \tag{2.1}$$

then

$$\lim_{n\to\infty} \frac{1}{A_n^{(\alpha)}} \sum_{\nu=0}^{n} A_{n-\nu}^{(\alpha-1)} |s_{k_\nu}(x) - f(x)|^k = 0$$

holds for any $\alpha > 0$ *and* $k > 0$, *almost everywhere in* (a, b), *for any increasing sequence* $\{k_\nu\}$.

TANDORI [4] has proved this theorem for $\alpha=1$ earlier.

In [2] we have generalized this theorem of TANDORI as follows:

Theorem C. *Under the hypothesis* (2. 1) *we have*

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{\nu=0}^{n}[s_{l_\nu}(x)-f(x)]^2=0$$

almost everywhere for any (non necessarily monotonic) sequence $\{l_\nu\}$ *of distinct non-negative integers.*

At the same time we proved the following

Theorem D. *Let* $\{a_n\}$ *be a given sequence of real numbers with* $\sum a_n^2<\infty$ *and*

$$na_n^2\geqq(n+1)a_{n+1}^2\qquad(n=1,2,\ldots).$$

If the orthogonal series (1. 1) *with* (1. 2) *is Abel-summable to* $f(x)$ *almost everywhere in* (a,b) *and*

$$c_n^2=O(a_n^2),$$

then we have

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{\nu=0}^{n}[\sigma_{l_\nu}^{\gamma-1}(x)-f(x)]^2=0$$

for any $\gamma>\frac{1}{2}$ *almost everywhere in* (a,b), *for any sequence* $\{l_\nu\}$ *of distinct non-negative integers.*

3. In the present note we intend to generalize further these theorems.

We consider a regular summation method T_n determined by a triangular matrix $\left\|\frac{\alpha_{nk}}{A_n}\right\|$ $\left(\alpha_{nk}\geqq0\text{ and }A_n=\sum_{k=0}^{n}\alpha_{nk}\right)$, i.e. if s_k tends to s, then

$$T_n=\frac{1}{A_n}\sum_{k=0}^{n}\alpha_{nk}s_k\to s.$$

Theorem 1. *Let* $k>0$. *If there exists a* $p>1$ *such that*

$$\frac{p}{p-1}k\geqq2\quad\text{and}\quad\left\{\sum_{\nu=1}^{n}\nu^{p-1}\alpha_{n\nu}^p\right\}^{1/p}\leqq K\sum_{\nu=1}^{n}\alpha_{n\nu}\tag{3. 1}$$

and if the series (1. 1) *with* (1. 2) *is* $(C,1)$*-summable to* $f(x)$ *almost everywhere in* (a,b), *then*

$$\lim_{n\to\infty}\frac{1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|\sigma_\nu^{\gamma-1}(x)-f(x)|^k=0\tag{3. 2}$$

almost everywhere in (a,b) *for any* $\gamma>\frac{1}{2}$.

It is clear that in the special case $\gamma=1$ and $\alpha_{n\nu}=A_{n-\nu}^{(\alpha-1)}$ $(\alpha>0)$ this theorem includes the Theorem A of SUNOUCHI; in fact,

$$\left\{\sum_{\nu=1}^{n}\nu^{p-1}(A_{n-\nu}^{(\alpha-1)})^p\right\}^{1/p}\leqq K_1\{n^{p-1}n^{(\alpha-1)p+1}\}^{1/p}=K_1n^\alpha\leqq K_2A_n^{(\alpha)}$$

for any $\alpha>0$ if p is near enough to 1.

It is easy to verify that in the cases

(3.3)
$$\alpha_{nk} = k^{\beta},\quad \beta > -1;\quad \alpha_{nk} = \frac{1}{k};\quad \alpha_{nk} = \frac{1}{k\log(k+2)};$$
$$\alpha_{nk} = \frac{1}{k\log(k+2)\log\log(k+4)}$$

and in those cases, which are similar to the above ones, the condition (3. 1) is satisfied for any $p>1$, consequently the statement (3. 2) holds for any $k>0$ in the cases mentioned above.

It follows easily from this theorem:

Theorem 2. *Let $k>0$. If there exists a $p>1$ such that the conditions* (3. 1) *holds and if*

(3.4)
$$\sum_{n=4}^{\infty} c_n^2 \log\log^2 n < \infty,$$

then we have

$$\lim_{n\to\infty} \frac{1}{A_n} \sum_{\nu=0}^{n} \alpha_{n\nu} |\sigma_\nu^{\gamma-1}(\{\mu_i\}; x) - f(x)|^k = 0$$

almost everywhere in (a, b) for any $\gamma > \frac{1}{2}$ and for any increasing sequence $\{\mu_i\}$; here we have set

$$\sigma_n^{\beta}(\{\mu_i\}; x) = \frac{1}{A_n^{(\beta)}} \sum_{\nu=0}^{n} A_{n-\nu}^{(\beta-1)} s_{\mu_\nu}(x).$$

Theorem 2 includes evidently the Theorem B of SUNOUCHI in the special case $\gamma=1$ and $\alpha_{n\nu}=A_{n-\nu}^{(\alpha-1)}$ $(\alpha>0)$.

Theorem 3. *Under the hypothesis of Theorem* 2 *we have*

$$\lim_{n\to\infty} \frac{1}{A_n} \sum_{\nu=0}^{n} \alpha_{n\nu} |s_{l_\nu}(x) - f(x)|^k = 0$$

almost everywhere in (a, b) for any sequence $\{l_\nu\}$ of distinct non-negative integers.

In particular, we have as

Corollary 1. *If the condition* (3. 4) *is satisfied, then*

$$\lim_{n\to\infty} \frac{1}{A_n^{(\alpha)}} \sum_{\nu=0}^{n} A_{n-\nu}^{(\alpha-1)} |s_{l_\nu}(x) - f(x)|^k = 0$$

holds for any $\alpha>0$ and $k>0$, almost everywhere in (a, b) for any sequence $\{l_\nu\}$ of distinct non-negative integers.

It is easy to see that this corollary generalizes the Theorems B and C.

Finally we prove the following

Theorem 4. *Let $\{d_n\}$ be a given real sequence with $\sum d_n^2 < \infty$ and*

(3.5)
$$n d_n^2 \geqq (n+1) d_{n+1}^2 \qquad (n=1, 2, \ldots),$$

further let $\gamma > \frac{1}{2}$ *and* $k > 0$. *If there exists a* $p > 1$ *such that the conditions* (3. 1) *hold, and if the series* (1. 1) *is* $(C, 1)$*-summable to* $f(x)$ *almost everywhere in* (a, b) *and, moreover,*

$$c_n^2 = O(d_n^2), \tag{3.6}$$

then

$$\lim_{n\to\infty} \frac{1}{A_n} \sum_{\nu=0}^{n} \alpha_{n\nu} |\sigma_{l_\nu}^{\gamma-1}(x) - f(x)|^k = 0$$

almost everywhere in (a, b) *for any sequence* $\{l_\nu\}$ *of distinct non-negative integers.*

This theorem includes the Theorem D in the special case $\alpha_{n\nu} = 1$ and $k = 2$, because the conditions (3. 1) are satisfied in the cases of (3. 3) for any $p > 1$, as we have seen it.

It seems worth while to observe also the following

Corollary 2. *Let* $\{d_n\}$ *be a given real sequence satisfying the conditions* $\sum d_n^2 < \infty$ *and* (3. 5). *If the series* (1. 1) *is* $(C, 1)$*-summable to* $f(x)$ *almost everywhere in* (a, b) *and* (3. 6) *is satisfied, then*

$$\lim_{n\to\infty} \frac{1}{A_n^{(\alpha)}} \sum_{\nu=0}^{n} A_{n-\nu}^{(\alpha-1)} |\sigma_{l_\nu}^{\gamma-1}(x) - f(x)|^k = 0$$

holds for any $\alpha > 0$, $k > 0$, *and* $\gamma > \frac{1}{2}$, *almost everywhere in* (a, b), *for any sequence* $\{l_\nu\}$ *of distinct non-negative integers.*

The method of proof of these theorems is that of Sunouchi [3] and of the author [2].

In the sequel, we use $K, K_1, K_2, \ldots$ to denote positive constants, not necessarily the same on any two occurences.

4. The following lemmas will be required for the proofs of the theorems.

Lemma 1. *Let* $\{\psi_k(x)\}$ $(k = 1, \ldots, N)$ *be an orthogonal system in* (a, b) *and let*

$$a_k^2 = \int_a^b \psi_k^2(x)\,dx \qquad (k = 1, 2, \ldots, N).$$

Then there exists a function $\delta(x)$ *such that*

$$|\psi_1(x) + \ldots + \psi_l(x)| \leqq \delta(x) \qquad (l = 1, 2, \ldots, N)$$

in (a, b) *and*

$$\int_a^b \delta^2(x)\,dx \leqq K_1 \log^2 N \sum_{k=1}^{N} a_k^2.$$

This Lemma is well known (cf. Kaczmarz—Steinhaus [1], p. 162).

Lemma 2. *If* $\sum_{n=0}^{\infty} c_n^2 < \infty$, *then*

$$\int_a^b \left\{ \sum_{n=1}^{\infty} n^{-1} |\sigma_n^{\alpha-1}(x) - \sigma_n^{\alpha}(x)|^2 \right\} dx \leqq K_2 \sum_{n=0}^{\infty} c_n^2 \qquad \left(\alpha > \frac{1}{2}\right).$$

This Lemma also is known (cf. [3], Lemma 1).

Lemma 3. *Let $k>0$ and $\sum c_n^2<\infty$. If there exists a $p>1$ such that the conditions* (3. 1) *are satisfied, then for $\gamma>\frac{1}{2}$ we have*

$$\int_a^b \left\{ \sup_{1\leqq n<\infty} \left(\frac{1}{A_n} \sum_{\nu=0}^{n} \alpha_{n\nu} |\sigma_\nu^{\gamma-1}(x)-\sigma_\nu^\gamma(x)|^k \right)^{1/k} \right\}^2 dx \leqq K_3 \sum_{n=0}^{\infty} c_n^2.$$

Proof. Applying Hölder's inequality, we obtain by (3. 1)

$$\frac{1}{A_n}\sum_{\nu=1}^{n} \alpha_{n\nu}|\sigma_\nu^{\gamma-1}(x)-\sigma_\nu^\gamma(x)|^k \leqq$$

$$\text{(4. 1)} \qquad \leqq \frac{1}{A_n}\left\{\sum_{\nu=1}^{n} \nu^{-1}|\sigma_\nu^{\gamma-1}(x)-\sigma_\nu^\gamma(x)|^{qk}\right\}^{1/q} \left\{\sum_{\nu=1}^{n} \nu^{p/q}\alpha_{n\nu}^p\right\}^{1/p} \leqq$$

$$\leqq K\left\{\sum_{\nu=1}^{n} \nu^{-1}|\sigma_\nu^{\gamma-1}(x)-\sigma_\nu^\gamma(x)|^{qk}\right\}^{1/q},$$

where $q=\dfrac{p}{p-1}$. Since $qk\geqq 2$, we have by Lemma 2 and (4. 1) that

$$\int_a^b \left\{ \sup_{1\leqq n<\infty} \left(\frac{1}{A_n} \sum_{\nu=1}^{n} \alpha_{n\nu} |\sigma_\nu^{\gamma-1}(x)-\sigma_\nu^\gamma(x)|^k \right)^{1/k} \right\}^2 dx \leqq$$

$$\leqq K_1 \int_a^b \left(\sum_{\nu=1}^{\infty} \nu^{-1}|\sigma_\nu^{\gamma-1}(x)-\sigma_\nu^\gamma(x)|^{qk}\right)^{2/qk} dx \leqq$$

$$\leqq K_1 \int_a^b \left(\sum_{\nu=1}^{\infty} \nu^{-1}|\sigma_\nu^{\gamma-1}(x)-\sigma_\nu^\gamma(x)|^{2}\right) dx \leqq K_2 \sum_{n=0}^{\infty} c_n^2.$$

5. Proof of Theorem 1. Since, by the hypothesis, the series (1. 1) is $(C, 1)$-summable, so the means $\sigma_n^\beta(x)$ $(\beta>0)$ converge to the function $f(x)$ almost everywhere in (a, b). From this fact it follows that in the following inequality

$$\frac{1}{A_n}\sum_{\nu=0}^{n} \alpha_{n\nu}|\sigma_\nu^{\gamma-1}(x)-f(x)|^k \leqq$$

$$\text{(5. 1)} \qquad \leqq \frac{K_1}{A_n}\sum_{\nu=0}^{n} \alpha_{n\nu}|\sigma_\nu^{\gamma-1}(x)-\sigma_\nu^\gamma(x)|^k + \frac{K_1}{A_n}\sum_{\nu=0}^{n} \alpha_{n\nu}|\sigma_\nu^{\gamma}(x)-f(x)|^k$$

the second sum tends to 0 almost everywhere in (a, b).

We shall show that the first sum tends to zero, too. To this effect, we choose N, for given $\varepsilon>0$, so that

$$\text{(5. 2)} \qquad \sum_{n\geqq N/4}^{\infty} c_n^2 < \varepsilon^3,$$

and we consider the series

$$(5.3)\qquad \sum_{n=1}^{\infty} a_n \varphi_n(x) \quad \text{with} \quad a_n = \begin{cases} c_n & \text{for} \quad n \leqq N, \\ 0 & \text{for} \quad n > N, \end{cases}$$

and

$$(5.4)\qquad \sum_{n=1}^{\infty} b_n \varphi_n(x) \quad \text{with} \quad b_n = \begin{cases} 0 & \text{for} \quad n \leqq N, \\ c_n & \text{for} \quad n > N. \end{cases}$$

Let us denote by $\sigma_\nu^\beta(a; x)$ and $\sigma_\nu^\beta(b; x)$, respectively, the ν-th Cesàro means of order β of the series (5. 3) and (5. 4). It is obvious that

$$(5.5)\qquad \sigma_\nu^\beta(x) = \sigma_\nu^\beta(a; x) + \sigma_\nu^\beta(b; x) \qquad (\nu = 1, 2, \ldots).$$

For the series (5. 3), the means

$$\frac{1}{A_n} \sum_{\nu=0}^{n} \alpha_{n\nu} |\sigma_\nu^{\gamma-1}(a; x) - \sigma_\nu^\gamma(a; x)|^k$$

converge clearly to zero almost everywhere. As to the series (5. 4), we obtain, using the Lemma 3 and (5. 2),

$$\int_a^b \left\{ \sup_{1 \leqq n < \infty} \left(\frac{1}{A_n} \sum_{\nu=0}^{n} \alpha_{n\nu} |\sigma_\nu^{\gamma-1}(b; x) - \sigma_\nu^\gamma(b; x)|^k \right)^{1/k} \right\}^2 \leqq K_1 \varepsilon^3.$$

Hence

$$\operatorname{meas} \left\{ x \,\middle|\, \limsup \left(\frac{1}{A_n} \sum_{\nu=0}^{n} \alpha_{n\nu} |\sigma_\nu^{\gamma-1}(b; x) - \sigma_\nu^\gamma(b; x)|^k \right)^{1/k} > \varepsilon \right\} \leqq K_1 \varepsilon.$$

That is, the means

$$\frac{1}{A_n} \sum_{\nu=0}^{n} \alpha_{n\nu} |\sigma_\nu^{\gamma-1}(b; x) - \sigma_\nu^\gamma(b; x)|^k$$

also converge to zero almost everywhere.

The statement (3. 2) follows from the above results by virtue of (5. 5).

This completes the proof of Theorem 1.

Proof of Theorem 2. We set

$$\gamma_n^2 = \sum_{k=\mu_{n-1}+1}^{\mu_n} c_k^2$$

and

$$\Phi_n(x) = \begin{cases} \dfrac{1}{\gamma_n} \displaystyle\sum_{k=\mu_{n-1}+1}^{\mu_n} c_k \varphi_k(x) & \text{for} \quad \gamma_n \neq 0, \\ \dfrac{1}{\sqrt{\mu_n - \mu_{n-1}}} \displaystyle\sum_{k=\mu_{n-1}+1}^{\mu_n} \varphi_k(x) & \text{for} \quad \gamma_n = 0. \end{cases}$$

By (3. 4),

$$\sum_{n=4}^{\infty} \gamma_n^2 \log\log^2 n = \sum_{n=4}^{\infty} \log\log^2 n \sum_{k=\mu_{n-1}+1}^{\mu_n} c_k^2 < \infty.$$

Hence, and from a well known theorem of KACZMARZ and MENSHOV, it follows hat the series

$$\sum_{n=1}^{\infty} \gamma_n \Phi_n(x)$$

is $(C, 1)$-summable to $f(x)$ almost everywhere in (a, b). Applying the Theorem 1 to the above series, we obtain the statement of Theorem 2.

Proof of Theorem 3. Under the condition (3. 4) the sequence $\{s_{2^m}(x)\}$ converges to $f(x)$ almost everywhere in (a, b). We write

$$C_m^2 = \sum_{n=2^m+1}^{2^{m+1}} c_n^2.$$

Let $m(\geqq 2)$ be any natural number, for which $C_m \neq 0$. Set $\mu_0(m) = 2^m$ and let $\mu_i(m)$ $(1 \leqq i \leqq N_m)$ be the smallest natural number for which

$$\sum_{n=\mu_{i-1}(m)+1}^{\mu_i(m)} c_n^2 \geqq \frac{C_m^2}{m} \quad \text{and} \quad \mu_i(m) \leqq 2^{m+1}$$

are valid. It is clear that $N_m \leqq m$. If $C_m = 0$, we write $\mu_0(m) = 2^m$ and $\mu_1(m) = 2^{m+1}$. Let us apply Lemma 1 to the functions

$$\psi_i^{(m)}(x) = s_{\mu_i(m)}(x) - s_{\mu_{i-1}(m)}(x) \qquad (1 \leqq i \leqq N_m).$$

Thus there exists a function $\delta_m(x)$ such that

$$(5.6) \qquad |s_{\mu_i(m)}(x) - s_{2^m}(x)| = \left| \sum_{j=1}^{i} \psi_j^{(m)}(x) \right| \leqq \delta_m(x) \qquad (1 \leqq i \leqq N_m)$$

in (a, b) and

$$\int_a^b \delta_m^2(x)\, dx \leqq K_1 \log^2 m \sum_{n=2^m+1}^{2^{m+1}} c_n^2 \leqq K_2 \sum_{n=2^m+1}^{2^{m+1}} c_n^2 \log\log^2 n.$$

Then, by (3. 4),

$$\sum_{m=2}^{\infty} \int_a^b \delta_m^2(x)\, dx > \infty$$

hence the series

$$\sum_{m=2}^{\infty} \delta_m^2(x)$$

converges almost everywhere. This gives by (5. 6) that

$$s_{\mu_i(m)}(x) - s_{2^m}(x) \to 0$$

for $m \to \infty$, almost everywhere in (a, b). Hence also $s_{\mu_i(m)}(x)$ $(m \to \infty)$ converges to the function $f(x)$ almost everywhere in (a, b).

Let us now define the following sequence of indices $\{\mu_\nu\}$: if $\mu_i(m) \leqq l_\nu < \mu_{i+1}(m)$ then set $\mu_\nu = \mu_i(m)$, and if $\mu_{N_m}(m) \leqq l_\nu < \mu_0(m+1)$ then $\mu_\nu = \mu_{N_m}(m)$.

It is easy to see that

$$
(5.7)\qquad \begin{aligned}
&\frac{1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|s_{l_\nu}(x)-f(x)|^k \leqq\\
&\qquad \leqq \frac{K_1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|s_{l_\nu}(x)-s_{\mu_\nu}(x)|^k+\frac{K_1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|s_{\mu_\nu}(x)-f(x)|^k.
\end{aligned}
$$

Since $s_{\mu_\nu}(x)\to f(x)$ $(\nu\to\infty)$, the second sum tends to 0 almost everywhere in (a, b).

From this point on, the proof runs similarly to the proof of the Theorem 1. Let us define N, $\{a_n\}$ and $\{b_n\}$ in the same way as under (5. 2), (5. 3) and (5. 4). Let us denote by $s_n(a; x)$ and $s_n(b; x)$, respectively, the n-th partial sums of series (5. 3) and (5. 4). It is evident that

$$
(5.8)\qquad s_n(x)=s_n(a;x)+s_n(b;x)\qquad (n=1, 2, \ldots).
$$

We can see easily that

$$
(5.9)\qquad \frac{1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|s_{l_\nu}(a;x)-s_{\mu_\nu}(a;x)|^k\to 0
$$

almost everywhere in (a, b). An analogous statement for the series (5. 4), can be obtained by the following easy computation. Using HÖLDER's inequality and (3. 1), we obtain

$$
\begin{aligned}
&\frac{1}{A_n}\sum_{\nu=1}^{n}\alpha_{n\nu}|s_{l_\nu}(b;x)-s_{\mu_\nu}(b;x)|^k \leqq\\
&\qquad \leqq \frac{1}{A_n}\left\{\sum_{\nu=1}^{n}\nu^{-1}|s_{l_\nu}(b;x)-s_{\mu_\nu}(b;x)|^{qk}\right\}^{1/q}\left\{\sum_{\nu=1}^{n}\nu^{p/q}\alpha_{n\nu}^{p}\right\}^{1/p}\leqq\\
&\qquad\qquad \leqq K\left\{\sum_{\nu=1}^{n}\nu^{-1}|s_{l_\nu}(b;x)-s_{\mu_\nu}(b;x)|^{qk}\right\}^{1/q}.
\end{aligned}
$$

Since $qk\geqq 2$, we have

$$
(5.10)\qquad \begin{aligned}
&\int_a^b\left\{\sup_{1\leqq n<\infty}\left(\frac{1}{A_n}\sum_{\nu=1}^{n}\alpha_{n\nu}|s_{l_\nu}(b;x)-s_{\mu_\nu}(b;x)|^k\right)^{1/k}\right\}^2 dx\leqq\\
&\qquad \leqq K_1\int_a^b\left(\sum_{\nu=1}^{\infty}\nu^{-1}|s_{l_\nu}(b;x)-s_{\mu_\nu}(b;x)|^{qk}\right)^{2/qk}dx\leqq\\
&\qquad\qquad \leqq K_1\int_a^b\left(\sum_{\nu=1}^{\infty}\nu^{-1}|s_{l_\nu}(b;x)-s_{\mu_\nu}(b;x)|^2\right)dx.
\end{aligned}
$$

An easy computation shows that *)

$$\sum_{\nu=1}^{\infty}\frac{1}{\nu}\int_a^b |s_{l_\nu}(b;x)-s_{\mu_\nu}(b;x)|^2\,dx=\sum_{\nu=1}^{\infty}\frac{1}{\nu}\sum_{k=\mu_\nu+1}^{l_\nu} b_k^2=$$

$$=\sum_{m=0}^{\infty}\sum_{2^m\leqq\mu_\nu<2^{m+1}}^{(\nu)}\frac{1}{\nu}\sum_{k=\mu_\nu+1}^{l_\nu} b_k^2\leqq\sum_{m=[\log N]}^{\infty}\Bigl(\sum_{2^m\leqq\mu_\nu<2^{m+1}}^{(\nu)}\frac{1}{\nu}\Bigr)\frac{C_m^2}{m}\leqq$$

$$\leqq\sum_{m=[\log N]}^{\infty}\Bigl(\sum_{\nu=1}^{2^m}\frac{1}{\nu}\Bigr)\frac{C_m^2}{m}\leqq K_2\sum_{k\geqq N/2}^{\infty}c_k^2.$$

From this and (5. 10) it follows

$$\int_a^b\Bigl\{\sup_{1\leqq n<\infty}\Bigl(\frac{1}{A_n}\sum_{\nu=1}^{n}\alpha_{n\nu}|s_{l_\nu}(b;x)-s_{\mu_\nu}(b;x)|^k\Bigr)^{1/k}\Bigr\}^2dx\leqq K_3\sum_{k\geqq N/2}^{\infty}c_k^2.$$

Hence

$$\operatorname{meas}\Bigl\{x\,\Big|\,\limsup\Bigl(\frac{1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|s_{l_\nu}(b;x)-s_{\mu_\nu}(b;x)|^k\Bigr)^{1/k}>\varepsilon\Bigr\}\leqq K_4\varepsilon.$$

From this we obtain that the means

$$\frac{1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|s_{l_\nu}(b;x)-s_{\mu_\nu}(b;x)|^k$$

converge to zero almost everywhere in (a, b).

Hence and from (5. 9) by (5. 8) we get that

$$\frac{1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|s_{l_\nu}(x)-s_{\mu_\nu}(x)|^k\to 0 \tag{5. 11}$$

almost everywhere.

Finally, from (5. 7) and (5. 11) we obtain the statement of Theorem 3.

Proof of Theorem 4. By the hypothesis of the theorem, the series (1. 1) is $(C, 1)$-summable to $f(x)$, thus

$$\lim_{n\to\infty}\frac{1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|\sigma_{l_\nu}^{\beta}(x)-f(x)|^k=0$$

almost everywhere in (a, b) for any $\beta>0$. Due to this fact, it suffices to prove the statement for the case $\frac{1}{2}<\gamma\leqq 1$. Since

$$\frac{1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|\sigma_{l_\nu}^{\gamma-1}(x)-f(x)|^k\leqq$$

$$\leqq\frac{K_1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|\sigma_{l_\nu}^{\gamma-1}(x)-\sigma_{l_\nu}^{\gamma}(x)|^k+\frac{K_1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|\sigma_{l_\nu}^{\gamma}(x)-f(x)|^k,$$

we only have to show that the first sum tends to zero.

*) $\Sigma^{(\nu)}$ denotes that the sum is taken for ν. We use the logarithm with basis 2.

For any positive ε, we choose N so that

$$\sum_{n=N}^{\infty} d_n^2 < \varepsilon^3. \tag{5.12}$$

We define further $\{a_n\}$ and $\{b_n\}$ in the same way as under (5. 3) and (5. 4). Let $\sigma_\nu^\beta(a; x)$ and $\sigma_\nu^\beta(b; x)$ have the same meaning as in the proof of Theorem 1.

It is easy to see that

$$\lim_{n\to\infty} \frac{1}{A_n} \sum_{\nu=0}^{n} \alpha_{n\nu} |\sigma_\nu^{\gamma-1}(a; x) - \sigma_\nu^\gamma(a; x)|^k = 0$$

almost everywhere in (a, b). The analogous statement for the series (5. 4) is the basis of the proof of this theorem. After a computation analogous to the proof of Lemma 3, we get

$$\begin{aligned} &\int_a^b \left\{ \sup_{1\le n<\infty} \left(\frac{1}{A_n} \sum_{\nu=1}^{n} \alpha_{n\nu} |\sigma_{l_\nu}^{\gamma-1}(b; x) - \sigma_{l_\nu}^\gamma(b; x)|^k \right)^{1/k} \right\}^2 dx \le \\ &\le K_1 \int_a^b \left(\sum_{\nu=1}^{\infty} \nu^{-1} |\sigma_{l_\nu}^{\gamma-1}(b; x) - \sigma_{l_\nu}^\gamma(b; x)|^2 \right) dx. \end{aligned} \tag{5.13}$$

An easy computation shows that

$$\begin{aligned} &\sum_{m=1}^{\infty} \frac{1}{m} \int_a^b |\sigma_{l_m}^{\gamma-1}(b; x) - \sigma_{l_m}^\gamma(b; x)|^2 \, dx \le \\ &\le K_1 \sum_{m=1}^{\infty} \frac{1}{m (A_{l_m}^{(\gamma)})^2} \sum_{k=1}^{l_m} (A_{l_m-k}^{(\gamma-1)})^2 k^2 b_k^2 \le K_2 \sum_{m=1}^{\infty} \frac{1}{m l_m^{2\gamma}} \sum_{k=1}^{l_m} (l_m - k + 1)^{2\gamma-2} k^2 b_k^2. \end{aligned} \tag{5.14}$$

Let us denote by m_i the ith natural number, for which $m_i \le l_{m_i}$, and by μ_n the nth natural number, for which $\mu_n > l_{\mu_n}$. Then we have

$$\begin{aligned} &\sum_{m=1}^{\infty} \frac{1}{m l_m^{2\gamma}} \sum_{k=1}^{l_m} (l_m - k + 1)^{2\gamma-2} k^2 b_k^2 = \\ &= \sum_{i=1}^{\infty} \frac{1}{m_i l_{m_i}^{2\gamma}} \sum_{k=1}^{l_{m_i}} (l_{m_i} - k + 1)^{2\gamma-2} k^2 b_k^2 + \sum_{n=1}^{\infty} \frac{1}{\mu_n l_{\mu_n}^{2\gamma}} \sum_{k=1}^{l_{\mu_n}} (l_{\mu_n} - k + 1)^{2\gamma-2} k^2 b_k^2. \end{aligned} \tag{5.15}$$

Since $l_{m_i} \ge m_i$, the first sum in (5. 15) is less than

$$K_3 \sum_{i=1}^{\infty} \frac{1}{m_i l_{m_i}^{2\gamma}} \left(\sum_{k=1}^{m_i-1} + \sum_{k=m_i}^{l_{m_i}} \right) (l_{m_i} - k + 1)^{2\gamma-2} k^2 b_k^2. \tag{5.16}$$

By virtue of (3. 5) and (3. 6), we have for $m_{i_0} \le N < m_{i_0+1}$

$$\begin{aligned} &\sum_{i=i_0}^{\infty} \frac{1}{m_i l_{m_i}^{2\gamma}} \sum_{k=\max(m_i, N)}^{l_{m_i}} (l_{m_i} - k + 1)^{2\gamma-2} k^2 d_k^2 \le \\ &\le \sum_{i=i_0}^{\infty} \frac{d_{\max(m_i, N)}^2}{l_{m_i}^{2\gamma}} \sum_{k=m_i}^{l_{m_i}} (l_{m_i} - k + 1)^{2\gamma-2} k. \end{aligned} \tag{5.17}$$

Since $\gamma > \frac{1}{2}$ and $l_{m_i} \geqq m_i$, it holds

$$\sum_{k=m_i}^{l_{m_i}} (l_{m_i} - k + 1)^{2\gamma-2}\, k \leqq \sum_{p=1}^{l_{m_i}-m_i+1} p^{2\gamma-2}(l_{m_i} - p + 1) \leqq K_4 l_{m_i}^{2\gamma}.$$

Hence and from (5. 17) it follows

$$\sum_{i=i_0}^{\infty} \frac{1}{m_i l_{m_i}^{2\gamma}} \sum_{k=\max(m_i N)}^{l_{m_i}} (l_{m_i} - k + 1)^{2\gamma-2} k^2 d_k^2 \leqq \tag{5. 18}$$
$$\leqq K_5 \left(d_N^2 + \sum_{i=i_0+1}^{\infty} d_{m_i}^2 \right) \leqq K_5 \sum_{j=N}^{\infty} d_j^2.$$

Let i_k be the least natural number for which $m_{i_k} > k$. Since $\frac{1}{2} < \gamma \leqq 1$ and $l_{m_i} \geqq m_i$ $(i = 1, 2, \ldots)$, it follows

$$\sum_{i=1}^{\infty} \frac{1}{m_i l_{m_i}^{2\gamma}} \sum_{k=1}^{m_i-1} (l_{m_i} - k + 1)^{2\gamma-2} k^2 b_k^2 \leqq \sum_{k=N+1}^{\infty} k^2 d_k^2 \sum_{i=i_k}^{\infty} \frac{(l_{m_i} - k + 1)^{2\gamma-2}}{m_i l_{m_i}^{2\gamma}} \leqq \tag{5. 19}$$
$$\leqq \sum_{k=N}^{\infty} k d_k^2 \sum_{i=i_k}^{\infty} \frac{(m_i - k + 1)^{2\gamma-2}}{m_i^{2\gamma}} \leqq \sum_{k=N}^{\infty} k d_k^2 \sum_{l=k}^{\infty} \frac{(l - k + 1)^{2\gamma-2}}{l^{2\gamma}} \leqq K_6 \sum_{k=N}^{\infty} d_k^2.$$

We can estimate the second sum under (5. 15) more easily than the first one. In fact, considering that $\mu_n > l_{\mu_n}$ $(n = 1, 2, \ldots)$, we have

$$\sum_{n=1}^{\infty} \frac{1}{\mu_n l_{\mu_n}^{2\gamma}} \sum_{k=1}^{l_{\mu_n}} (l_{\mu_n} - k + 1)^{2\gamma-2} k^2 b_k^2 \leqq K_1 \sum_{k=N}^{\infty} k^2 d_k^2 \sum_{l_{\mu_n} \geqq k} \frac{(l_{\mu_n} - k + 1)^{2\gamma-2}}{\mu_n l_{\mu_n}^{2\gamma}} \leqq$$
$$\leqq K_1 \sum_{k=N}^{\infty} k d_k^2 \sum_{l=k}^{\infty} \frac{(l - k + 1)^{2\gamma-2}}{l^{2\gamma}} \leqq K_2 \sum_{k=N}^{\infty} d_k^2.$$

Hence and from (5. 13)—(5. 19), considering (5. 12), we obtain that

$$\int_a^b \left\{ \sup_{1 \leqq n < \infty} \left(\frac{1}{A_n} \sum_{\nu=1}^{n} \alpha_{n\nu} |\sigma_{l_\nu}^{\gamma-1}(b; x) - \sigma_{l_\nu}^{\gamma}(b; x)|^k \right)^{1/k} \right\}^2 dx \leqq K_3 \sum_{k=N}^{\infty} d_k^2 < \varepsilon^3. \tag{5. 20}$$

The proof runs similarly to the proof of Theorem 3. From (5. 20) it follows that

$$\text{meas} \left\{ x \,\middle|\, \limsup \left(\frac{1}{A_n} \sum_{\nu=1}^{n} \alpha_{n\nu} |\sigma_{l_\nu}^{\gamma-1}(b; x) - \sigma_{l_\nu}^{\gamma}(b; x)|^k \right)^{1/k} > \varepsilon \right\} \leqq K_1 \varepsilon,$$

i. e.

$$\lim_{n\to\infty} \frac{1}{A_n} \sum_{\nu=1}^{n} \alpha_{n\nu} |\sigma_{l_\nu}^{\gamma-1}(b; x) - \sigma_{l_\nu}^{\gamma}(b; x)|^k = 0$$

almost everywhere in (a, b).

This completes the proof of Theorem 4.

Literature

[1] S. KACZMARZ—H. STEINHAUS, *Theorie der Orthogonalreihen* (Warszawa—Lwów, 1935).
[2] L. LEINDLER, Über die starke Summierbarkeit der Orthogonalreihen, *Acta Sci. Math.*, **23** (1962), 82—89.
[3] G. SUNOUCHI, On the strong summability of orthogonal series, *Acta Sci. Math.*, **27** (1966), 71—76.
[4] K. TANDORI, Über die orthogonalen Funktionen. VI (Eine genaue Bedingung für die starke Summation), *Acta Sci. Math.*, **20** (1959), 14—18.

(Received April 30, 1966)

Über charakteristische Eigenschaften der Divisionsringe[1])

Von KARLHEINZ BAUMGARTNER in Gießen

In der vorliegenden Note werden von SZELE herrührende Ergebnisse diskutiert bzw. verschärft. Es gilt:

(1) Ein Ring ohne Nullteiler mit wenigstens einem minimalen Linksideal ist ein Divisionsring (vgl. [3]).

(2) Besitzt ein kommutativer Ring R ein minimales Ideal M, so ist der Restklassenring von R nach einem M nicht enthaltenden Primideal ein Körper. Daher sind alle Primideale, die mit einem minimalen Ideal nur das Nullelement gemeinsam haben auch maximal und modular.[2])

Bemerkung. Das ist eine Richtigstellung der zweiten Aussage in [3]. Man kann nämlich nicht schließen, daß mit R auch jeder Restklassenring nach einem Primideal ein minimales Ideal besitzt. Z.B. ist im Ring $R = \mathbf{Z} \oplus GF(2)$ offensichtlich der Primkörper $GF(2)$ ein Primideal und gleichzeitig das einzige minimale Ideal, wenn $\mathbf{Z}$ der Ring der ganzen Zahlen ist. Aber $R/GF(2) \cong \mathbf{Z}$ besitzt keine minimalen Ideale. Man sieht auch, daß das Primideal $GF(2)$ keineswegs maximal in R ist.

Die Aussage (1) ist eine Verschärfung der Tatsache, daß ein Artinring ohne Nullteiler ein Divisionsring ist[3]). Hingegen ist (2) keine Verschärfung der ebenfalls bekannten Tatsache, daß in einem kommutativen Artinring jedes Primideal maximal und modular ist.[3])

Hier wird gezeigt, daß (1) noch wesentlich verschärft werden kann. Nämlich zu:

(I) Besitzt ein Ring R wenigstens ein minimales Linksideal und enthält dieses wenigstens ein Element a, welches kein Rechtsnullteiler von R ist, so ist R ein Divisionsring.

Da (2) eine unmittelbare Folge aus (1) folgt, liegt es nahe auch davon die Verallgemeinerung auszusprechen:

(II) Es sei A ein Ideal des Ringes R. Gibt es nun wenigstens ein A minimal umfaßendes Linksideal B und gibt es wenigstens ein b aus B, so daß stets mit $r \cdot b$ auch r Element von A ist, so ist A maximal und modular.

Wir bemerken, daß (II) eine unmittelbare Folge von (I) ist. Bildet man nämlich R/A, B/A, so folgt, daß B/A in R/A minimales Linksideal mit nicht lauter R/A-

[1]) Divisionsringe nennen wir die Körper und Schiefkörper.

[2]) Ein Linksideal L eines Ringes R heißt modular, wenn es ein e aus R gibt, sodaß für jedes a aus R stets $a - ae$ in L ist.

[3]) Vgl. B. L. v. D. WAERDEN, *Algebra.* II, S. 198, oder E. ARTIN, C. J. NESBITT, R. M. THRALL, *Rings with minimum condition,* 1946, S. 59, Theorem 6. 10.

Rechtsnullteilern ist. Nach (I) ist dann R/A ein Divisionsring, also ist A modular und maximal in R.

Daß die Bedingung nur hinreichend ist, sieht man mit Hilfe von (I) schon am Beispiel der einfachen Artinringe. Hingegen ist sie bekanntlich im kommutativen Fall wohl auch notwendig. Klar ist, daß sich (II) für Artinringe vereinfacht und daß offenbar (2) eine Folge von (II) ist.

Sei nun M ein R-Linksmodul, W eine wohlgeordnete Menge der Ordnungszahl I erzeugender Elemente von M.[4]) Faßt man W als $I\times 1$ Spalte auf und bezeichnet A_I den vollen Matrixring der zeilenendlichen $I\times I$ Matrizen über R, so bilden die Matrizen C aus A_I mit $C\cdot W=0$ ein Linksideal L (Relationslinksideal) in A_I. Ist $N(L)$ der Normalisator von L in A_I, d.h. der größte Unterring, in dem L sogar Ideal ist und bedeutet K die Menge der zeilenendlichen Matrizen B aus $N(L)$ mit $A_I\cdot B\subset L$, so haben wir

Lemma. *Für den R-Endomorphismenring E von M gilt die Isomorphie*

$$E\cong N(L)/K.$$

Beweis. Ist B aus $N(L)$, so wird durch β: $W\to B\cdot W$ in M ein R-Endomorphismus induziert. Denn: Durch die geforderte Additivität und Linearität bestimmt β eine Abbildung von M in sich mit diesen Eigenschaften. Zu jedem m aus M gibt es eine $1\times I$ Zeile A mit Elementen aus R, sodaß $m=A\cdot W$ gilt. Ist nun $A\cdot W=0$, so folgt wegen $B\in N(L)$

$$\beta(AW)=A(\beta W)=A(BW)=(AB)W=0.$$

Da jeder R-Endomorphismus durch ein $B\in A_I$ beschrieben werden kann und weil dabei das Nullelement von M natürlich festbleibt, folgt $L\cdot B\subset L$, also ist $B\in N(L)$. Somit wird $N(L)$ auf E epimorph abgebildet. Aus der Definition von K folgt, daß K der Epimorphiekern ist. Damit ist der Beweis beendet.

Da E stets den identischen Endomorphismus enthält ist $K\supset L$ von A_I verschieden. Daher ist klar:

Korollar. (Jacobson [1], S. 26.) *Ist M ein irreduzibler R-Modul, so gilt für seinen Endomorphismendivisionsring E die Isomorphie $E\cong N(L)/L$. Dabei ist L das annulierende (modulare maximale) Linksideal* (0: u) *eines von Null verschiedenen Elements u aus M. Umgekehrt gilt für jedes maximale modulare Linksideal L, daß $N(L)/L$ ein Divisionsring ist.*

Die unter (I) gemachte Behauptung folgt nun aus der Tatsache, daß einerseits der R-Endomorphismenring eines minimalen Linksideals (aufgefaßt als irreduzibler R-Linksmodul) ein Divisionsring E ist, andererseits ist das annullierende Linksideal L des Elements a das Nullideal, da ja a kein Rechtsnullteiler ist. Dann ist $N(0)=R$ und somit $R\cong E$, also ein Divisionsring.

Schlußbemerkung. Ähnlich könnte man zeigen, daß ein Ring ohne Linksideale (vgl. [2], [4]) ein Divisionsring oder ein Zeroring von Primzahlordnung ist.

[4]) Das soll heißen: Jedes $m\in M$ hat eine Darstellung $m=\sum_{i\in I} r_i w_i$ mit $r_i\in R$, $w_i\in W$, wobei höchstens endlich viele r_i von Null verschieden sind.

Übrigens ist dies auch aus (I) zu erschließen. Der Ring R ist nämlich selbst ein minimales Linksideal und ist er kein Zeroring (von Primzahlordnung), so gibt es ein $a \in R$ welches kein (Rechts)-Annullator ist. Somit muß das a annullierende Linksideal L das Nullideal sein. Also ist a kein Rechtsnullteiler und (I) anwendbar.

Literatur

[1] N. JACOBSON, *Structure of Rings* (New Haven, 1956).
[2] F. A. SZÁSZ, Note on rings in which every proper left-ideal is cyclic, *Fund. Math.*, **44** (1957), 330—332.
[3] T. SZELE, Eine kennzeichnende Eigenschaft der Schiefkörper, *Comment. Math. Helv.*, **22** (1949), 115—116.
[4] T. SZELE, Die Ringe ohne Linksideale, *Acad. Rep. Pop. Român. Bul. Şti.*, **1** (1949), 788—790.

(Eingegangen am 1. April 1966)

Ergodic type theorems in von Neumann algebras

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Introduction

Let $\mathbf{A}$ be a von Neumann algebra[1]) in a complex Hilbert space $\mathfrak{H}$, and let $\mathcal{G}$ be a group of automorphisms of $\mathbf{A}$ [2]). Denote by $\mathbf{A}^{\mathcal{G}}$ the set of all elements of $\mathbf{A}$ which are invariant with respect to each element of $\mathcal{G}$. Taking into account the algebraic and topological properties of the elements of $\mathcal{G}$ ([13], chap. I, § 4, Th. 2, Cor. 1), one can see easily that $\mathbf{A}^{\mathcal{G}}$ is a von Neumann subalgebra of $\mathbf{A}$. For any $T \in \mathbf{A}$, let $\mathcal{K}_0(T, \mathcal{G})$ denote the smallest convex subset of $\mathbf{A}$ which contains the *orbit* of T *under* $\mathcal{G}$ [3]). Let $\mathcal{K}(T, \mathcal{G})$ be the weak closure of $\mathcal{K}_0(T, \mathcal{G})$ [4]). The investigations concerning the center-valued trace theory of von Neumann algebras and the results of some other works (for example [1], [2], [7]) naturally give the idea of seeking conditions on $\mathbf{A}$ and $\mathcal{G}$ under which the set $\mathcal{K}(T, \mathcal{G})$ meets $\mathbf{A}^{\mathcal{G}}$ for every $T \in \mathbf{A}$.

The purpose of this paper is to give a sufficient condition in order that $\mathcal{K}(T, \mathcal{G}) \cap \mathbf{A}^{\mathcal{G}}$ consist of exactly one element for every $T \in \mathbf{A}$ (Theorem 1.) This is the subject of § 2. The next § 3 is devoted to establishing under this condition a mapping of $\mathbf{A}$ onto $\mathbf{A}^{\mathcal{G}}$ which reminds us, from many points of view, of the Dixmier trace $\natural$ of a finite von Neumann algebra (Theorem 2). In § 4, some simple consequences of the above results are given. § 1 contains preliminary results and examples.

The main results of this paper were announced in [5], with the proof of Theorem 1 in a less detailed form.

§ 1

First of all let us set down some notations.

If $\mathbf{A}$ is a von Neumann algebra and $\mathcal{G}$ is a group of automorphisms of $\mathbf{A}$, denote by $\mathcal{R}(\mathbf{A}, \mathcal{G})$ the set of all ultra-weakly continuous linear forms on $\mathbf{A}$ which

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*) This author's contribution to the paper was done while he was a Postdoctorate Fellow at Queen's University in Kingston, of the National Research Council of Canada.

[1]) For the theory of von Neumann algabras, cf. [3]. The terminology of [3] will be freely used in the following.

[2]) By an automorphism of a von Neumann algebra, we always mean a $*$-automorphism.

[3]) By the orbit of T under $\mathcal{G}$ we mean the set of the elements $\{\theta(T)\}_{\theta \in \mathcal{G}}$.

[4]) For a given pair $(\mathbf{A}, \mathcal{G})$, the notations $\mathcal{K}_0(T, \mathcal{G})$, $\mathcal{K}(T, \mathcal{G})$ $(T \in \mathbf{A})$ will be permanently used by us, without explaining again what they mean.

are *invariant with respect to* $\mathcal{G}$ (that is if $\sigma\in\mathcal{R}(\mathbf{A},\mathcal{G})$ then for every $T\in\mathbf{A}$ and $\theta\in\mathcal{G}$ we have $\sigma(\theta(T))=\sigma(T)$). Let $\mathcal{R}^+(\mathbf{A},\mathcal{G})$ denote the set of all positive elements of $\mathcal{R}(\mathbf{A},\mathcal{G})$. For any element σ of $\mathcal{R}^+(\mathbf{A},\mathcal{G})$, E_σ will denote the *support* of σ ([3], chap. I, § 4, Def. 3). It is easy to see that $E_\sigma\in\mathbf{A}^{\mathcal{G}}$. The group of all inner automorphisms of $\mathbf{A}$ will be denoted by $\mathcal{I}(\mathbf{A})$.

With these notations we have the following

Proposition 1. *Let* $\mathbf{A}$ *be a von Neumann algebra in a complex Hilbert space* $\mathfrak{H}$, *and let* $\mathcal{G}$ *be a group of automorphisms of* $\mathbf{A}$. *The following four conditions are equivalent:*

(i) *For every* $T\in\mathbf{A}^+$ [5]), $T\neq 0$ *there exists an element* σ *of* $\mathcal{R}^+(\mathbf{A},\mathcal{G})$ *such that* $\sigma(T)\neq 0$;

(ii) *For every* $T\in(\mathbf{A}^{\mathcal{G}})^+$, $T\neq 0$ *there exists an element* σ *of* $\mathcal{R}^+(\mathbf{A},\mathcal{G})$ *with* $\sigma(T)\neq 0$;

(iii) *There exists a family* $\{\sigma_\iota\}_{\iota\in I}$ *of elements of* $\mathcal{R}^+(\mathbf{A},\mathcal{G})$ *such that* $E_{\sigma_\iota}E_{\sigma_\varkappa}=0$ *for* $\iota\neq\varkappa$ *and* $\sum_{\iota\in I}E_{\sigma_\iota}=I_{\mathfrak{H}}$. [6])

(iv) $\sup_{\sigma\in\mathcal{R}^+(\mathbf{A},\mathcal{G})} E_\sigma=I_{\mathfrak{H}}$.

Proof. (i)$\Rightarrow$(ii) is evident.

(ii)$\Rightarrow$(iii). In fact, let $\{\sigma_\iota\}_{\iota\in I}$ be a maximal family of elements of $\mathcal{R}^+(\mathbf{A},\mathcal{G})$ such that $E_{\sigma_\iota}E_{\sigma_\varkappa}=0$ for $\iota\neq\varkappa$. Such a family exists by the ZORN's lemma. Set $E=\sum_{\iota\in I}E_{\sigma_\iota}$, and prove that $E=I_{\mathfrak{H}}$. To do this, suppose the contrary that is that $E\neq I_{\mathfrak{H}}$. Put $F=I_{\mathfrak{H}}-E$. Since $F\in(\mathbf{A}^{\mathcal{G}})^+$, $F\neq 0$, in virtue of (ii), there exists an element σ of $\mathcal{R}^+(\mathbf{A},\mathcal{G})$ such that $\sigma(F)\neq 0$. Set $\sigma'(T)=\sigma(FTF)$ for every $T\in\mathbf{A}$. As $F\in\mathbf{A}^{\mathcal{G}}$, we obtain that $\sigma'\in\mathcal{R}^+(\mathbf{A},\mathcal{G})$. Furthermore, we have $\sigma'\neq 0$ and $\sigma'(E)=0$. This means that $E_{\sigma'}\neq 0$ and $E_{\sigma'}\leqq F$, and this contradicts the maximality of the family $\{\sigma_\iota\}_{\iota\in I}$.

(iii)$\Rightarrow$(iv) is evident.

(iv)$\Rightarrow$(i). Suppose that (i) is not true. Then there exists an element $T\in\mathbf{A}^+$, $T\neq 0$ such that $\sigma(T)=0$ for every $\sigma\in\mathcal{R}^+(\mathbf{A},\mathcal{G})$. This means that $E_\sigma TE_\sigma=0$ for every $\sigma\in\mathcal{R}^+(\mathbf{A},\mathcal{G})$. Thus for every $x\in\mathfrak{H}$ we get $\|T^{\frac{1}{2}}E_\sigma x\|=0$, i.e. $T^{\frac{1}{2}}E_\sigma=0$. As, by (iv), $\sup_{\sigma\in\mathcal{R}^+(\mathbf{A},\mathcal{G})} E_\sigma=I_{\mathfrak{H}}$, we obtain that $T^{\frac{1}{2}}=0$, that is $T=0$ which is impossible, and this completes the proof of Proposition 1.

Definition 1. Let $\mathbf{A}$ be a von Neumann algebra and let $\mathcal{G}$ be a group of automorphisms of $\mathbf{A}$. $\mathbf{A}$ is said to be *finite with respect to* $\mathcal{G}$ (or $\mathcal{G}$*-finite*) if $\mathbf{A}$ and $\mathcal{G}$ satisfy any of the equivalent conditions of Proposition 1.

Remarks. 1. To say that $\mathbf{A}$ is $\mathcal{I}(\mathbf{A})$-finite is equivalent to say that $\mathbf{A}$ is finite in the usual sense of the global theory of the von Neumann algebras ([3], chap. I, § 6, Def. 5).

2. If $\mathbf{A}$ is $\mathcal{G}$-finite then $\mathbf{A}$ is finite with respect to any subgroup of $\mathcal{G}$.

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[5]) For a von Neumann algebra $\mathbf{A}$, $\mathbf{A}^+$ denotes the set of all non-negative self-adjoint elements of $\mathbf{A}$.

[6]) $I_{\mathfrak{H}}$ denotes the identity operator of the Hilbert space $\mathfrak{H}$.

Now let us give *examples* for pairs $(\mathbf{A}, \mathscr{G})$ such that $\mathbf{A}$ is $\mathscr{G}$-finite.

1. $\mathbf{A}$ *is a finite von Neumann algebra and* $\mathscr{G}$ *is an arbitrary subgroup of* $\mathscr{I}(\mathbf{A})$.

2. $\mathbf{A}$ *is a finite factor and* $\mathscr{G}$ *is an arbitrary group of automorphisms of* $\mathbf{A}$. In fact, if $\mathrm{Tr}\,(\cdot)$ is the canonical trace of $\mathbf{A}$ ([3], chap. III, no. 4) and θ is an arbitrary element of $\mathscr{G}$ then $\varphi(T)=\mathrm{Tr}\,(\theta(T))$ $(T\in\mathbf{A})$ is also a normalized trace[7]) on $\mathbf{A}$. Therefore, for every $T\in\mathbf{A}$ we have $\mathrm{Tr}\,(T)=\varphi(T)=\mathrm{Tr}\,(\theta(T))$ ([3], chap. I. § 6, Th. 3, Cor.), and this means that $\mathrm{Tr}\,(\cdot)\in\mathscr{R}^+(\mathbf{A}, \mathscr{G})$. Since $\mathrm{Tr}\,(\cdot)$ is a strictly positive linear form on $\mathbf{A}$, we obtain that $\mathbf{A}$ is $\mathscr{G}$-finite.

3. Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be von Neumann algebras in the Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively. Let $\mathscr{G}_i$ be a group of automorphisms of $\mathbf{A}_i$ for every $i=1, 2$. Put $\mathfrak{H}=\mathfrak{H}_1\otimes\mathfrak{H}_2$ and $\mathbf{A}=\mathbf{A}_1\otimes\mathbf{A}_2$. If $\theta_1\in\mathscr{G}_1$ and $\theta_2\in\mathscr{G}_2$, there exists a uniquely defined automorphism θ of $\mathbf{A}$ such that $\theta(T_1\otimes T_2)=\theta_1(T_1)\otimes\theta_2(T_2)$ for every $T_1\in\mathbf{A}_1$ and $T_2\in\mathbf{A}_2$ ([3], chap. I, § 4. Prop. 2). Denote by $\mathscr{G}_1\otimes\mathscr{G}_2$ the set of all θ obtained from all possible pairs $\{\theta_1\in\mathscr{G}_1, \theta_2\in\mathscr{G}_2\}$ in this way. Under the usual multiplication, $\mathscr{G}=\mathscr{G}_1\otimes\mathscr{G}_2$ is a group of automorphisms of $\mathbf{A}$.

Proposition 2. *If* $\mathbf{A}_1$ *is* $\mathscr{G}_1$*-finite and* $\mathbf{A}_2$ *is* $\mathscr{G}_2$*-finite then* $\mathbf{A}$ *is* $\mathscr{G}$*-finite.*

Proof. In virtue of Definition 1, it is enough to show that $\sup\limits_{\sigma\in\mathscr{R}^+(\mathbf{A},\mathscr{G})} E_\sigma=I_{\mathfrak{H}}$. To do this, consider an arbitrary element $\sigma_i\in\mathscr{R}^+(\mathbf{A}_i, \mathscr{G}_i)$ $(i=1, 2)$. It is known ([3], chap. I, § 4, Th. 1) that for each $i=1, 2$, there exists a sequence $\{x_k^{(i)}\}_{k=1}^\infty$ of elements of $\mathfrak{H}_i$ with $\sum\limits_{k=1}^\infty \|x_k^{(i)}\|^2<+\infty$ such that for every $T_i\in\mathbf{A}_i$ we have

$$\sigma_i(T_i)=\sum_{k=1}^\infty (T_i x_k^{(i)}\,|\,x_k^{(i)}).$$

Now for every $T\in\mathbf{A}$, put

$$\sigma(T)=\sum_{k=1}^\infty\sum_{l=1}^\infty (T[x_k^{(1)}\otimes x_l^{(2)}]\,|\,x_k^{(1)}\otimes x_l^{(2)}).$$

It is easy to see that $\sigma(T_1\otimes T_2)=\sigma_1(T_1)\sigma_2(T_2)$ for every $T_1\in\mathbf{A}_1, T_2\in\mathbf{A}_2$. By linearity and continuity, from this we can conclude that $\sigma\in\mathscr{R}^+(\mathbf{A}, \mathscr{G})$. Furthermore,

$$E_{\sigma_1}\mathfrak{H}_1=\mathfrak{X}^{\mathbf{A}_1'}_{\{x_k^{(1)}\}_{k=1}^\infty},\qquad E_{\sigma_2}\mathfrak{H}_2=\mathfrak{X}^{\mathbf{A}_2'}_{\{x_k^{(2)}\}_{k=1}^\infty}\quad\text{and}\quad E_\sigma\mathfrak{H}=\mathfrak{X}^{\mathbf{A}'}_{\{x_k^{(1)}\otimes x_l^{(2)}\}_{k,l=1}^\infty}\ {}^{8)}$$

([3], chap. I, § 4, no. 6).

On the other hand, we have $\mathbf{A}_1'\otimes\mathbf{A}_2'\subseteqq\mathbf{A}'$. This implies that

$$E_{\sigma_1}\otimes E_{\sigma_2}\leqq E_\sigma. \tag{1.1}$$

Since $\mathbf{A}_1$ and $\mathbf{A}_2$ are $\mathscr{G}_1$- and $\mathscr{G}_2$-finite, respectively, we have that

$$\sup_{\sigma_1\in\mathscr{R}^+(\mathbf{A}_1,\mathscr{G}_1),\ \sigma_2\in\mathscr{R}^+(\mathbf{A}_2,\mathscr{G}_2)} E_{\sigma_1}\otimes E_{\sigma_2}=I_{\mathfrak{H}}.$$

This together with (1.1) gives that $\sup\limits_{\sigma\in\mathscr{R}^+(\mathbf{A},\mathscr{G})} E_\sigma=I_{\mathfrak{H}}$, and so the proof of Proposition 2 is complete.[9])

[7]) That is, $\varphi(I_{\mathfrak{H}})=1$.

[8]) For these notations, cf. [3], chap. I, § 1, no. 4.

[9]) For this reasoning, see [3], chap. I, § 4, Ex. 6.

Proposition 2 enables us to give examples for pairs $(\mathbf{A}, \mathcal{G})$ such that $\mathbf{A}$ is purely infinite ([3], chap. I, § 6, Def. 5), $\mathcal{G}$ is a non-trivial group of automorphisms[10]) of $\mathbf{A}$, and $\mathbf{A}$ is $\mathcal{G}$-finite. For instance, let $\mathbf{M}_1$ be a finite factor, and let $\mathcal{G}_1$ be an arbitrary but non-trivial group of automorphisms of $\mathbf{M}_1$. Let $\mathbf{M}_2$ be a purely infinite von Neumann algebra. Then $\mathbf{A}=\mathbf{M}_1\otimes\mathbf{M}_2$ is purely infinite ([6]). Put $\mathcal{G}=\mathcal{G}_1\otimes\mathcal{I}$, where $\mathcal{I}$ is the trivial group of automorphisms of $\mathbf{M}_2$. Then $\mathcal{G}$ is a non-trivial group of automorphisms of $\mathbf{A}$ and $\mathbf{A}$ is $\mathcal{G}$-finite (cf. Ex. 2 above and Prop. 2).

§ 2

Our main result can be stated as follows.

Theorem 1. *Let $\mathbf{A}$ be a von Neumann algebra and let $\mathcal{G}$ be a group of automorphisms of $\mathbf{A}$. Suppose that $\mathbf{A}$ is $\mathcal{G}$-finite. Then for every $T\in\mathbf{A}$, $\mathcal{K}(T,\mathcal{G})\cap\mathbf{A}^{\mathcal{G}}$ consists of exactly one element.*

A key-role in the proof of this theorem is played by the ergodic theorem of Alaoglu and Birkhoff ([4], Th. 1. 1. 3.). For convenience, we recall the reader just for a particular part of it we need.

Lemma 1. *Let $\mathfrak{H}$ be a complex Hilbert space, and let $\mathcal{U}$ be a group of unitary operators in $\mathfrak{H}$. For an arbitrary $x\in\mathfrak{H}$, denote by $c(x,\mathcal{U})$ the smallest convex subset of $\mathfrak{H}$ which contains the orbit of x under $\mathcal{U}$. Let $\bar{c}(x,\mathcal{U})$ be the closure of $c(x,\mathcal{U})$ in $\mathfrak{H}$. Then there exists a unique element x_0 in $\bar{c}(x,\mathcal{U})$ such that $Ux_0=x_0$ for every $U\in\mathcal{U}$. The mapping $x\to x_0$ is linear.*

Proof of Theorem 1. Let T be an arbitrary but fixed element of $\mathbf{A}$, and consider an arbitrary σ in $\mathcal{R}^+(\mathbf{A},\mathcal{G})$. As σ is ultra-weakly continuous,

$$\mathfrak{m}_\sigma=\{S\in\mathbf{A}\colon \sigma(S^*S)=0\}$$

is an ultra-weakly closed left ideal of A. Consider the quotient vector space $\mathbf{A}/\mathfrak{m}_\sigma$, and let $S\to\eta_\sigma(S)$ denote the canonical mapping of $\mathbf{A}$ onto $\mathbf{A}/\mathfrak{m}_\sigma$. For every $R, S\in\mathbf{A}$, set

$$\langle\eta_\sigma(R)|\eta_\sigma(S)\rangle_\sigma=\sigma(S^*R). \tag{2. 1}$$

Then the vector space $\mathbf{A}/\mathfrak{m}_\sigma$ becomes a pre-Hilbert space with respect to the inner product (2. 1). Let $\mathfrak{H}_\sigma$ be the completion of $\mathbf{A}/\mathfrak{m}_\sigma$ in the norm defined by (2. 1).[11]) Now, let θ be an arbitrary element of $\mathcal{G}$. For any $\eta_\sigma(S)\in\mathbf{A}/\mathfrak{m}_\sigma$ $(S\in\mathbf{A})$, put

$$\overset{(\sigma)}{\theta_0}\,\eta_\sigma(S)=\eta_\sigma(\theta(S)). \tag{2.2}$$

First of all we note that $\overset{(\sigma)}{\theta_0}$ is uniquely defined, that is its definition does not depend on the special choice of the representatives of the elements of $\mathbf{A}/\mathfrak{m}_\sigma$. Indeed,

[10]) That is $\mathcal{G}$ does not consists just of the identical automorphism of $\mathbf{A}$.

[11]) For this construction, see [3], chap. I, § 4, no. 1.

since σ is invariant with respect to θ, θ sends $\mathfrak{m}_\sigma$ onto itself. So, if S_1 and S_2 are two elements of $\mathbf{A}$ such that $\eta_\sigma(S_1)=\eta_\sigma(S_2)$ then $S_1-S_2\in\mathfrak{m}_\sigma$ and

$$\overset{(\sigma)}{\theta}_0\eta_\sigma(S_1)-\overset{(\sigma)}{\theta}_0\eta_\sigma(S_2)=\eta_\sigma(\theta(S_1))-\eta_\sigma(\theta(S_2))=\eta_\sigma(\theta(S_1-S_2))=0,$$

which means that $\overset{(\sigma)}{\theta}_0\,\eta_\sigma(S_1)=\overset{(\sigma)}{\theta}_0\,\eta_\sigma(S_2)$. It is clear that $\overset{(\sigma)}{\theta}_0$ is linear. Furthermore, $\overset{(\sigma)}{\theta}_0\,(\mathbf{A}/\mathfrak{m}_\sigma)\subseteqq\mathbf{A}/\mathfrak{m}_\sigma$ by definition. Now, if $\eta_\sigma(S)$ is an arbitrary element of $\mathbf{A}/\mathfrak{m}_\sigma$, then $\overset{(\sigma)}{\theta}_0\,\eta_\sigma(\theta^{-1}(S))=\eta_\sigma(S)$ which means that $\overset{(\sigma)}{\theta}_0$ is surjective.

Consider now two arbitrary elements S_1 and S_2 of $\mathbf{A}$. Then we have

$$\begin{aligned}\langle\overset{(\sigma)}{\theta}_0\eta_\sigma(S_1)\,|\,\overset{(\sigma)}{\theta}_0\eta_\sigma(S_2)\rangle_\sigma&=\sigma(\theta(S_2^*)\theta(S_1))=\sigma(\theta(S_2^*S_1))=\\&=\sigma(S_2^*S_1)=\langle\eta_\sigma(S_1)\,|\,\eta_\sigma(S_2)\rangle_\sigma.\end{aligned}\tag{2.3}$$

Therefore, $\overset{(\sigma)}{\theta}_0$ can be uniquely extended to a unitary operator $\overset{(\sigma)}{\theta}$ of $\mathfrak{H}_\sigma$. Furthermore, it is not hard to prove that $[\overset{(\sigma)}{\theta}]^*=(\theta^{-1})^{(\sigma)}$, and that the family $\{\overset{(\sigma)}{\theta}\}_{\theta\in\mathscr{G}}$ is a group under the usual multiplication of unitary operators. Denote this group by $\overset{(\sigma)}{\mathscr{G}}$. Now, applying Lemma 1 to $\mathfrak{H}_\sigma$ and $\overset{(\sigma)}{\mathscr{G}}$, we obtain a unique point, say $\overset{(\sigma)}{x}$, in $\bar{c}(\eta_\sigma(T),\overset{(\sigma)}{\mathscr{G}})$ such that

$$\overset{(\sigma)}{\theta}\,\overset{(\sigma)}{x}=\overset{(\sigma)}{x}\tag{2.4}$$

for every $\overset{(\sigma)}{\theta}\in\overset{(\sigma)}{\mathscr{G}}$. We are going to prove that $\overset{(\sigma)}{x}\in\mathbf{A}/\mathfrak{m}_\sigma$. To do this, consider a sequence $\{x_n\}_{n=1}^\infty$ of elements of $c(\eta_\sigma(T),\overset{(\sigma)}{\mathscr{G}})$ with $\|x_n-\overset{(\sigma)}{x}\|_\sigma\to0$ if $n\to\infty$. Let $\{T_n\}_{n=1}^\infty$ be a sequence of elements of $\mathscr{K}_0(T,\mathscr{G})$ such that $\eta_\sigma(T_n)=x_n$ for every $n=1,2,\dots$. Then we have

$$\sigma((T_m-T_n)^*(T_m-T_n))=\|\eta_\sigma(T_m)-\eta_\sigma(T_n)\|_\sigma^2=\|x_m-x_n\|_\sigma^2\to0\tag{2.5}$$

for $m,n\to\infty$. As $\|T_m-T_n\|\leqq2\|T\|$ [12]), in virtue of [3], chap. I, § 4, Prop. 4, we conclude from (2. 5) that $(T_m-T_n)E_\sigma\to0$ strongly for $m,n\to\infty$. Therefore, there exists a well-defined element S_1 of A such that

$$T_nE_\sigma\to S_1\tag{2.6}$$

strongly for $n\to\infty$. Now, as $\|T_nE_\sigma-S_1\|\leqq2\|T\|$ $(n=1,2,\dots)$, using again the proposition of [3] which has just been quoted, we obtain that

$$\|x_n-\eta_\sigma(S_1)\|_\sigma^2=\|\eta_\sigma(T_n)-\eta_\sigma(S_1)\|_\sigma^2=\sigma((T_n-S_1)^*(T_n-S_1))\to0\tag{2.7}$$

for $m,n\to\infty$. So,

$$\overset{(\sigma)}{x}=\eta_\sigma(S_1)\quad\text{with}\quad S_1\in\mathbf{A},\tag{2.8}$$

[12]) $\|\ \|$ denotes the usual norm of bounded linear operators.

that is

$$\overset{(\sigma)}{x} \in \mathbf{A}/\mathfrak{m}_\sigma. \tag{2.9}$$

As the ultra-weak topology is compatible with the vector space structure of $\mathbf{A}$ and $\mathfrak{m}_\sigma$ is ultra-weakly closed, the set $\overset{-1}{\eta_\sigma}(\overset{(\sigma)}{x})$ is ultra-weakly closed in $\mathbf{A}$. Set

$$\mathbf{A}_\sigma^t(T) = \overset{-1}{\eta_\sigma}(\overset{(\sigma)}{x}) \cap \mathbf{A}_t \tag{2.10}$$

where $t = \|T\|$ and $\mathbf{A}_t = \{S \in \mathbf{A}: \|S\| \leqq t\}$. Then $\mathbf{A}_\sigma^t(T)$ is weakly closed as the weak topology coincides with the ultra-weak one on norm-bounded parts of $\mathbf{A}$. Furthermore, $\mathbf{A}_\sigma^t(T)$ is not empty as it contains at least S_1 constructed above (see (2. 8)). As a next step of our proof, let us construct the set $\mathbf{A}_\sigma^t(T)$ for every $\sigma \in \mathscr{R}^+(\mathbf{A}, \mathscr{G})$. Then, if $\sigma_1, \sigma_2 \in \mathscr{R}^+(\mathbf{A}, \mathscr{G})$, we have

$$\mathbf{A}_{\sigma_1+\sigma_2}^t(T) \subseteqq \mathbf{A}_{\sigma_i}^t(T) \qquad (i=1, 2). \tag{2.11}$$

Since $\sigma_1 + \sigma_2 \in \mathscr{R}^+(\mathbf{A}, \mathscr{G})$ and $\sigma_1 + \sigma_2 \geqq \sigma_i$ [13]) $(i=1, 2)$, to prove (2. 11) we have to show that if $\sigma', \sigma'' \in \mathscr{R}^+(\mathbf{A}, \mathscr{G})$ with $\sigma' \leqq \sigma''$ then $\mathbf{A}_{\sigma''}^t(T) \subseteqq \mathbf{A}_{\sigma'}^t(T)$. Well, suppose that we are given σ', σ'' from $\mathscr{R}^+(\mathbf{A}, \mathscr{G})$ with $\sigma' \leqq \sigma''$, and take an arbitrary element S of $\mathbf{A}_{\sigma''}^t(T)$. We have to prove that $S \in \mathbf{A}_{\sigma'}^t(T)$. First we note that $S \in \mathbf{A}_{\sigma''}^t(T)$ implies $\|S\| \leqq t$. So to show that $S \in \mathbf{A}_{\sigma'}^t(T)$, it suffices to prove that $\eta_{\sigma'}(S) = \overset{(\sigma')}{x}$ (where $\overset{(\sigma')}{x}$ plays the same role in the case of σ' as $\overset{(\sigma)}{x}$ did in the case of σ). Let $\{T_n\}_{n=1}^\infty$ be a sequence of elements of $\mathscr{K}_0(T, \mathscr{G})$ such that

$$\|\eta_{\sigma''}(T_n) - \overset{(\sigma'')}{x}\|_{\sigma''} \to 0 \qquad (n \to \infty).$$

By our assumption, $S \in \mathbf{A}_{\sigma''}^t(T)$ that is $\eta_{\sigma''}(S) = \overset{(\sigma'')}{x}$. Therefore, we have

$$\|\eta_{\sigma'}(T_n) - \eta_{\sigma'}(S)\|_{\sigma'}^2 = \sigma'\big((T_n - S)^*(T_n - S)\big) \leqq$$

$$\leqq \sigma''\big((T_n - S)^*(T_n - S)\big) = \|\eta_{\sigma''}(T_n) - \eta_{\sigma''}(S)\|_{\sigma''}^2 = \|\eta_{\sigma''}(T_n) - \overset{(\sigma'')}{x}\|_{\sigma''}^2 \to 0$$

if $n \to \infty$. So we obtain that $\eta_{\sigma'}(S) \in \bar{c}(\eta_{\sigma'}(T), \overset{(\sigma')}{\mathscr{G}})$, and it remains to prove that $\eta_{\sigma'}(S)$ is invariant with respect to each element of $\overset{(\sigma')}{\mathscr{G}}$. Let $\overset{(\sigma')}{\theta} \in \overset{(\sigma')}{\mathscr{G}}$ be arbitrary. Then

$$\|\overset{(\sigma')}{\theta} \eta_{\sigma'}(S) - \eta_{\sigma'}(R)\|_{\sigma'} = \|\eta_{\sigma'}(\theta(S)) - \eta_{\sigma'}(S)\|_{\sigma'} \leqq$$

$$\leqq \|\eta_{\sigma''}(\theta(S)) - \eta_{\sigma''}(S)\|_{\sigma''} = \|\overset{(\sigma'')}{\theta} \overset{(\sigma'')}{x} - \overset{(\sigma'')}{x}\|_{\sigma''} = 0.$$

So $\overset{(\sigma')}{\theta} \eta_{\sigma'}(S) = \eta_{\sigma'}(S)$ for every $\overset{(\sigma')}{\theta} \in \overset{(\sigma')}{\mathscr{G}}$. Using the uniqueness of $\overset{(\sigma')}{x}$ in $\bar{c}(\eta_{\sigma'}(T), \overset{(\sigma')}{\mathscr{G}})$ we get that $\eta_{\sigma'}(S) = \overset{(\sigma')}{x}$, indeed. Hence (2. 11) is proved. In virtue of (2. 11), the

[13]) That means that $\sigma_1(T) + \sigma_2(T) \geqq \sigma_i(T)$ $(i=1, 2)$ for every $T \in \mathbf{A}^+$.

amily $\{\mathbf{A}^t_\sigma(T)\}_{\sigma\in\mathscr{R}^+(\mathbf{A},\mathscr{G})}$ is a filter basis on $\mathbf{A}_t$. It is known that $\mathbf{A}_t$ is weakly ompact ([3], chap. I, § 3, Th. 2). Thus, as each $\mathbf{A}^t_\sigma(T)$ is weakly closed, we obtain that

(2. 12) $$\mathbf{A}^t(T) = \bigcap_{\sigma\in\mathscr{R}^+(\mathbf{A},\mathscr{G})} \mathbf{A}^t_\sigma(T) \neq \emptyset.$$

Now put

(2. 13) $$\mathbf{A}_\sigma(T) = \overset{-1}{\eta_\sigma}(\overset{(\sigma)}{x})$$

for every $\sigma\in\mathscr{R}^+(\mathbf{A},\mathscr{G})$. Then

(2. 14) $$\mathbf{A}(T) = \bigcap_{\sigma\in\mathscr{R}^+(\mathbf{A},\mathscr{G})} \mathbf{A}_\sigma(T)$$

is not empty since $\mathbf{A}^t_\sigma(T)\subseteqq\mathbf{A}_\sigma(T)$ for every $\sigma\in\mathscr{R}^+(\mathbf{A},\mathscr{G})$ and (2. 12) holds. Now if $S_1\in\mathbf{A}(T)$ and $S_2\in\mathbf{A}(T)$, then for every $\sigma\in\mathscr{R}^+(\mathbf{A},\mathscr{G})$ we obtain that

$$\eta_\sigma(S_1) = \eta_\sigma(S_2) = \overset{(\sigma)}{x},$$

hence $\sigma\big((S_1-S_2)^*(S_1-S_2)\big)=0$. As $\mathbf{A}$ is supposed to be $\mathscr{G}$-finite, we get that $S_1=S_2$. This means that $\mathbf{A}(T)=\mathbf{A}^t(T)$, and it consists of exactly one element. Denote this unique element by $T^{\mathscr{G}}$. We are going to show that

(2. 14) $$\mathscr{K}(T,\mathscr{G})\cap\mathbf{A}^{\mathscr{G}} = \{T^{\mathscr{G}}\},$$

where $\{T^{\mathscr{G}}\}$ denotes the set consisting of the element $T^{\mathscr{G}}$ alone. To do this, consider an arbitrary element θ of $\mathscr{G}$. For every $\sigma\in\mathscr{R}^+(\mathbf{A},\mathscr{G})$ we have

$$\sigma\big((\theta(T^{\mathscr{G}})-T^{\mathscr{G}})^*(\theta(T^{\mathscr{G}})-T^{\mathscr{G}})\big) = \|\eta_\sigma\big(\theta(T^{\mathscr{G}})\big)-\eta_\sigma(T^{\mathscr{G}})\|^2_\sigma =$$
$$= \|\overset{(\sigma)}{\theta}\,\overset{(\sigma)}{x}-\overset{(\sigma)}{x}\|^2_\sigma = 0.$$

Hence $\theta(T^{\mathscr{G}})=T^{\mathscr{G}}$ which gives

(2. 15) $$T^{\mathscr{G}}\in\mathbf{A}^{\mathscr{G}}.$$

Now let $x_1,\dots,x_n$ be an arbitrary finite family of elements of $\mathfrak{H}$. Then there exists an element σ_0 of $\mathscr{R}^+(\mathbf{A},\mathscr{G})$ such that $E_{\sigma_0}x_i=x_i$ for every $i=1,\dots,n$. In fact, consider a family $\{\sigma_\iota\}_{\iota\in I}$ of elements of $\mathscr{R}^+(\mathbf{A},\mathscr{G})$ with $\sigma_\iota(I_{\mathfrak{H}})=1$ $(\iota\in I)$, $E_{\sigma_\iota}E_{\sigma_\varkappa}=0$ for $\iota\neq\varkappa$, and $\sum_{\iota\in I}E_{\sigma_\iota}=I_{\mathfrak{H}}$. Then there exists a countable subfamily $\{\sigma_{\iota_n}\}_{n=1}^\infty$ of $\{\sigma_\iota\}_{\iota\in I}$ such that $\left(\sum_{n=1}^\infty E_{\sigma_{\iota_n}}\right)x_i=x_i$ $(i=1,\dots,n)$. For every $T\in\mathbf{A}$ put

$$\sigma_0(T) = \sum_{n=1}^\infty \frac{1}{2^n}\sigma_{\iota_n}(T).$$

It is clear that $\sigma_0\in\mathscr{R}^+(\mathbf{A},\mathscr{G})$ ([3], chap. I, § 3, no. 3). Furthermore, if for a projection P of $\mathbf{A}$ we have $\sigma_0(P)=0$, then $\sigma_{\iota_n}(P)=0$ for every $n=1,2,\dots$. This means that $\sum_{n=1}^\infty E_{\sigma_{\iota_n}}\leqq E_{\sigma_0}$. On the other hand,

$$\sigma_0\left(E_{\sigma_0}-\sum_{n=1}^\infty E_{\sigma_{\iota_n}}\right) = \sum_{n=1}^\infty\frac{1}{2^n}[\sigma_{\iota_n}(E_{\sigma_0})-\sigma_{\iota_n}(E_{\sigma_{\iota_n}})] =$$
$$= \sum_{n=1}^\infty\frac{1}{2^n}[\sigma_{\iota_n}(E_{\sigma_{\iota_n}})-\sigma_{\iota_n}(E_{\sigma_{\iota_n}})] = 0.$$

From this it follows that $E_{\sigma_0} - \sum_{n=1}^{\infty} E_{\sigma_{\iota_n}} \leqq I - E_{\sigma_0}$, which gives that $E_{\sigma_0} - \sum_{n=1}^{\infty} E_{\sigma_{\iota_n}} = 0$. So $E_{\sigma_0} = \sum_{n=1}^{\infty} E_{\sigma_{\iota_n}}$, that is $E_{\sigma_0} x_i = x_i$ $(i = 1, 2, \ldots, n)$. Now let $\{T_m\}_{m=1}^{\infty}$ be a sequence of elements of $\mathscr{K}_0(T, \mathscr{G})$ such that $\|\eta_{\sigma_0}(T_m) - \eta_{\sigma_0}(T^{\mathscr{G}})\|_{\sigma_0} \to 0$ for $m \to \infty$. This implies that

$$(T_m - T^{\mathscr{G}}) E_{\sigma_0} \to 0$$

strongly for $m \to \infty$ ([3], chap. I, § 4, Prop 4). Thus, for every $\varepsilon > 0$ there exists an index $m_0 = m_0(\varepsilon)$ such that

$$\|(T_{m_0} - T^{\mathscr{G}}) E_{\sigma_0} x_i\| < \varepsilon \qquad (i = 1, \ldots, n).$$

As $E_{\sigma_0} x_i = x_i$ $(i = 1, \ldots, n)$, we get that

$$\|(T_{m_0} - T^{\mathscr{G}}) x_i\| < \varepsilon \qquad (i = 1, \ldots, n).$$

Hence, $T^{\mathscr{G}} \in \mathscr{K}(T, \mathscr{G})$, as the strong closure and the weak closure of $\mathscr{K}_0(T, \mathscr{G})$ coincide ([3], chap. I, § 3, Th. 1). Thus we have proved that

$$\{T^{\mathscr{G}}\} \subseteqq \mathscr{K}(T, \mathscr{G}) \cap \mathbf{A}^{\mathscr{G}}. \tag{2. 16}$$

Now let S be an arbitrary element of $\mathscr{K}(T, \mathscr{G}) \cap A^{\mathscr{G}}$. Then using again [3], chap. I, § 4, Prop. 4, it is not hard to see that for every $\sigma \in \mathscr{R}^+(\mathbf{A}, \mathscr{G})$ we have $\eta_\sigma(S) \in \bar{c}(\eta_\sigma(T), \overset{(\sigma)}{\mathscr{G}})$ and $\eta_\sigma(S)$ is invariant with respect to the elements of $\overset{(\sigma)}{\mathscr{G}}$. Therefore, we have $\eta_\sigma(S) = \overset{(\sigma)}{x}$ for every $\sigma \in \mathscr{R}^+(\mathbf{A}, \mathscr{G})$. Hence we obtain that $S \in A(T) = \{T^{\mathscr{G}}\}$, that is

$$\mathscr{K}(T, \mathscr{G}) \cap \mathbf{A}^{\mathscr{G}} \subseteqq \{T^{\mathscr{G}}\}, \tag{2. 17}$$

which implies, together with (2. 16), that

$$\{T^{\mathscr{G}}\} = \mathscr{K}(T, \mathscr{G}) \cap \mathbf{A}^{\mathscr{G}}. \tag{2. 18}$$

Since T was arbitrary in $\mathbf{A}$, Theorem 1 is completely proved.

§ 3

Now we are in the position to prove

Theorem 2. *Let* $\mathbf{A}$ *be a von Neumann algebra in a complex Hilbert space* $\mathfrak{H}$, *and let* $\mathscr{G}$ *be a group of automorphisms of* $\mathbf{A}$. *Suppose that* $\mathbf{A}$ *is* $\mathscr{G}$*-finite. Then the mapping* $T \to T^{\mathscr{G}}$ [14]) *possesses the following properties:*

(i) *for every* $\sigma \in \mathscr{R}(\mathbf{A}, \mathscr{G})$ *and* $T \in \mathbf{A}$ *we have* $\sigma(T) = \sigma(T^{\mathscr{G}})$;

(ii) $T \to T^{\mathscr{G}}$ *is linear and strictly positive*;[15])

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[14]) $T^{\mathscr{G}}$, as above, denotes the unique element of $\mathscr{K}(T, \mathscr{G}) \cap \mathbf{A}^{\mathscr{G}}$ (cf. Th. 1).

[15]) In general, if $T \to \Phi(T)$ is a mapping of A into itself, Φ is said to be *positive* if $T \in \mathbf{A}^+$ implies $\Phi(T) \in \mathbf{A}^+$. Φ is *strictly positive*, if $T \in \mathbf{A}^+$, $T \neq O$ imply $\Phi(T) \geqq O$, $\Phi(T) \neq O$.

(iii) *if* $T\in\mathbf{A}$, $S\in\mathbf{A}^{\mathscr{G}}$ *we have* $(ST)^{\mathscr{G}}=ST^{\mathscr{G}}$ *and* $(TS)^{\mathscr{G}}=T^{\mathscr{G}}S$;
(iv) $T\to T^{\mathscr{G}}$ *is ultra-weakly and ultra-strongly continuous*;
(v) *for every* $T\in\mathbf{A}^{\mathscr{G}}$ *we have* $T=T^{\mathscr{G}}$;
(vi) $(\theta(T))^{\mathscr{G}}=T^{\mathscr{G}}$ *for every* $T\in\mathbf{A}$ *and* $\theta\in\mathscr{G}$.

Conversely, if we do not suppose that $\mathbf{A}$ *is* $\mathscr{G}$*-finite but we know that there exists an ultra-weakly continuous positive linear mapping* $T\to T'$ *of* $\mathbf{A}$ *onto* $\mathbf{A}^{\mathscr{G}}$ *such that*
a) $T=T'$ *for every* $T\in\mathbf{A}^{\mathscr{G}}$,
b) $(\theta(T))'=T$ *for every* $T\in\mathbf{A}, \theta\in\mathscr{G}$,
then $\mathbf{A}$ *is necessarily* $\mathscr{G}$*-finite and for every* $T\in\mathbf{A}$ *we have* $T'=T^{\mathscr{G}}$ *(cf.*[14]*)*.

Proof. (i) It suffices to take into account the construction of $T^{\mathscr{G}}$ and to note that if $\sigma\in\mathscr{R}(\mathbf{A},\mathscr{G})$ then σ is weakly continuous on every norm-bounded part of $\mathbf{A}$, in particular on $\mathscr{K}(T,\mathscr{G})$.

(ii) Consider two arbitrary elements S and T of $\mathbf{A}$. Then we have $S^{\mathscr{G}}+T^{\mathscr{G}}\in\mathbf{A}^{\mathscr{G}}$. We are going to prove that $S^{\mathscr{G}}+T^{\mathscr{G}}$ belongs to $\mathscr{K}(S+T,\mathscr{G})$, too. According to the notations used in the proof of Theorem 1, for every $\sigma\in\mathscr{R}^{+}(\mathbf{A},\mathscr{G})$, $\eta_\sigma(S^{\mathscr{G}})$ is the fixed point of $\bar{c}(\eta_\sigma(S),\overset{(\sigma)}{\mathscr{G}})$ and $\eta_\sigma(T^{\mathscr{G}})$ is the fixed point of $\bar{c}(\eta_\sigma(T),\overset{(\sigma)}{\mathscr{G}})$, given by Lemma 1. In virtue of the second assertion of this lemma, $\eta_\sigma(S^{\mathscr{G}})+\eta_\sigma(T^{\mathscr{G}})=$ $=\eta_\sigma(S^{\mathscr{G}}+T^{\mathscr{G}})$ is the fixed point of $\bar{c}(\eta_\sigma(S)+\eta_\sigma(T),\overset{(\sigma)}{\mathscr{G}})=\bar{c}(\eta_\sigma(S+T),\overset{(\sigma)}{\mathscr{G}})$ for every $\sigma\in\mathscr{R}^{+}(\mathbf{A},\mathscr{G})$. This means that $S^{\mathscr{G}}+T^{\mathscr{G}}\in\mathbf{A}(S+T)=\mathscr{K}(S+T,\mathscr{G})\cap\mathbf{A}^{\mathscr{G}}$. Thus $S^{\mathscr{G}}+T^{\mathscr{G}}=(S+T)^{\mathscr{G}}$. It is evident that $T\to T^{\mathscr{G}}$ is homogenous. Now if $T\in\mathbf{A}^{+}$, then $T^{\mathscr{G}}\geqq 0$ as $T^{\mathscr{G}}\in\mathscr{K}(T,\mathscr{G})\subseteqq\mathbf{A}^{+}$. If $T\in\mathbf{A}^{+}$ and $T\neq O$, then $T^{\mathscr{G}}\neq O$. Indeed, if $T^{\mathscr{G}}=O$ then, in virtue of (i), we have $\sigma(T)=\sigma(T^{\mathscr{G}})=0$ for every $\sigma\in\mathscr{R}^{+}(\mathbf{A},\mathscr{G})$. Since $\mathbf{A}$ is $\mathscr{G}$-finite, from this it follows $T=O$, which completes the proof of (ii).

(iii) follows easily from the construction of the mapping $T\to T^{\mathscr{G}}$.

(iv) First we prove that the mapping $T\to T^{\mathscr{G}}$ is *normal* that is if $\{T_\iota\}_{\iota\in I}$ is an upward directed family of elements of $\mathbf{A}^{+}$ with $\sup\limits_{\iota\in I} T_\iota=T$, then $\sup\limits_{\iota\in I} T_\iota^{\mathscr{G}}=T^{\mathscr{G}}$ holds. In fact, since $T\to T^{\mathscr{G}}$ is positive, $\{T_\iota^{\mathscr{G}}\}$ is an upward directed family of $(\mathbf{A}^{\mathscr{G}})^{+}$ and $T_\iota^{\mathscr{G}}\leqq T^{\mathscr{G}}$ $(\iota\in I)$. Put $S=\sup_{\iota\in I} T_\iota^{\mathscr{G}}$. Then $S\in\mathbf{A}^{\mathscr{G}}$ ([3], App. II.), and $S\leqq T^{\mathscr{G}}$. In virtue of (i), for every $\sigma\in\mathscr{R}^{+}(\mathbf{A},\mathscr{G})$ we obtain that

$$\sigma(T^{\mathscr{G}}-S)=\sigma(T^{\mathscr{G}})-\sigma(S)=\sigma(T)-\sup_{\iota\in I}\sigma(T_\iota^{\mathscr{G}})=$$
$$=\sigma(T)-\sup_{\iota\in I}\sigma(T_\iota)=\sigma(T)-\sigma(T)=0.$$

So $T^{\mathscr{G}}=S=\sup T_\iota^{\mathscr{G}}$. From this it follows that $T\to T^{\mathscr{G}}$ is ultra-weakly continuous ([3], chap. I, § 4, Th. 2). Furthermore, for every $T\in\mathbf{A}$ we obtain

$$O\leqq[(T-T^{\mathscr{G}})^{*}(T-T^{\mathscr{G}})]^{\mathscr{G}}=(T^{*}T)^{\mathscr{G}}-T^{*\mathscr{G}}T^{\mathscr{G}}-T^{*\mathscr{G}}T^{\mathscr{G}}+T^{*\mathscr{G}}T^{\mathscr{G}}=$$
$$=(T^{*}T)^{\mathscr{G}}-T^{*\mathscr{G}}T^{\mathscr{G}}$$

(cf. (ii) and (iii)). Thus $T^{*\mathscr{G}}T^{\mathscr{G}}\leqq(T^{*}T)^{\mathscr{G}}$, and this gives that $T\to T^{\mathscr{G}}$ is ultra-strongly continuous as well ([3]. chap. I, § 4, Th. 2).

(v) is evident.

(vi) is a consequence of the fact that $\mathscr{K}(\theta(T),\mathscr{G})=\mathscr{K}(T,\mathscr{G})$ for every $T\in\mathbf{A}$. Hence the first part of Theorem 2 is proved.

As far as the second part of Theorem 2 is concerned, we can proceed as follows. Let T_0 be an arbitrary element of $(\mathbf{A}^{\mathcal{G}})^+$ such that $T_0 \neq O$. Then there exists an element x of $\mathfrak{H}$ such that $(T_0 x|x) > 0$. For every $T \in \mathbf{A}$ put

$$\sigma(T) = (T'x|x). \tag{3. 1}$$

By our hypotheses on the mapping $T \to T'$, one can easily see that $\sigma \in \mathcal{R}^+(\mathbf{A}, \mathcal{G})$ with $\sigma(T_0) \neq 0$. Thus, in virtue of Definition 1, $\mathbf{A}$ is $\mathcal{G}$-finite. Furthermore, if $T \in \mathbf{A}$, then for every $S \in \mathcal{K}_0(T, \mathcal{G})$ we get that $S' = T'$ (cf. especially hypothesis b) in Theorem 2). As $T \to T'$ is supposed to be ultra-weakly continuous, the same holds for every $S \in \mathcal{K}(T, \mathcal{G})$. In particular $T' = (T^{\mathcal{G}})' = T^{\mathcal{G}}$, which completes the proof of Theorem 2.

Definition 2. If the von Neumann algebra $\mathbf{A}$ is finite with respect to a group $\mathcal{G}$ of its automorphisms, then the mapping $T \to T^{\mathcal{G}}$ given in Theorem 2 is called the *$\mathcal{G}$-canonical mapping* of $\mathbf{A}$.

§ 4

1. Let us give some direct consequences of the results of §§ 2—3.

Proposition 3. *Let $\mathbf{A}$ be a von Neumann algebra, and let $\mathcal{G}$ be a group of automorphisms of $\mathbf{A}$. Suppose that $\mathbf{A}$ is $\mathcal{G}$-finite. If $\sigma_1, \sigma_2 \in \mathcal{R}(\mathbf{A}, \mathcal{G})$ are such that, for every $T \in \mathbf{A}^{\mathcal{G}}$, $\sigma_1(T) = \sigma_2(T)$ holds, then $\sigma_1 = \sigma_2$.*

Proof. If $T \in \mathbf{A}$ then

$$\sigma_1(T) = \sigma_1(T^{\mathcal{G}}) = \sigma_2(T^{\mathcal{G}}) = \sigma_2(T)$$

(cf. Theorem 2, (i)), where $T \to T^{\mathcal{G}}$ is the $\mathcal{G}$-canonical mapping of $\mathbf{A}$, and this proves Proposition 3.

In the following for a given pair $(\mathbf{A}, \mathcal{G})$, $\mathcal{R}(\mathbf{A}^{\mathcal{G}})$ will denote the set of all ultra-weakly continuous linear forms on $\mathbf{A}^{\mathcal{G}}$. Then under the same condition on $\mathbf{A}$ and $\mathcal{G}$ as in Proposition 3, we have

Corollary 1. *Every element σ_0 of $\mathcal{R}(\mathbf{A}^{\mathcal{G}})$ can be uniquely extended to an element σ of $\mathcal{R}(\mathbf{A}, \mathcal{G})$.*

Proof. For any $T \in \mathbf{A}$, put

$$\sigma(T) = \sigma_0(T^{\mathcal{G}}).$$

Then σ evidently belongs to $\mathcal{R}(\mathbf{A}, \mathcal{G})$ (cf. Theorem 2). The uniqueness of the extension follows now from Proposition 3.

Without making any restriction on $\mathbf{A}$ and $\mathcal{G}$ we can conclude from Proposition 3 also the following

Corollary 2. *If $\sigma_1, \sigma_2 \in \mathcal{R}^+(\mathbf{A}, \mathcal{G})$ with $\sigma_1(T) = \sigma_2(T)$ for every $T \in \mathbf{A}^{\mathcal{G}}$, then $\sigma_1 = \sigma_2$.*

Proof. Consider the projection $E = \sup(E_{\sigma_1}, E_{\sigma_2})$. It is evident that $E \in \mathbf{A}^{\mathcal{G}}$. Consider the von Neumann algebra $\mathbf{A}_E$ ([3], chap. I, § 1, no. 2). Then $\mathcal{G}$ canonically induces a group of automorphisms $\mathcal{G}_E$ of $\mathbf{A}_E$, and the restrictions σ_{1_E} and σ_{2_E} of σ_1 and σ_2 to $\mathbf{A}_E$, respectively, belong to $\mathcal{R}^+(\mathbf{A}_E, \mathcal{G}_E)$. Hence $\mathbf{A}_E$ is $\mathcal{G}_E$-finite. Furthermore, for every $T_E \in (\mathbf{A}_E)^{\mathscr{G}_E}$ we have $\sigma_{1_E}(T_E) = \sigma_{2_E}(T_E)$. So, in virtue of Proposition 3, $\sigma_{1_E} = \sigma_{2_E}$. Therefore, if $T \in \mathbf{A}$, then $\sigma_1(ETE) = \sigma_{1_E}(T_E) = \sigma_{2_E}(T_E) = \sigma_2(ETE)$. On the other hand, since $\sigma_i(T) = \sigma_i(ETE)$ $(i = 1, 2)$ for every $T \in \mathbf{A}$, we can conclude that $\sigma_1 = \sigma_2$, which proves Corollary 2.

Proposition 4. *Let* $\mathbf{A}$ *be a von Neumann algebra in a Hilbert space* $\mathfrak{H}$, *and let* $\mathscr{G}_1$ *and* $\mathscr{G}_2$ *be two groups of automorphisms of* $\mathbf{A}$. *Suppose that* $\mathbf{A}$ *is* $\mathscr{G}_1$*-finite, and suppose that for every* $\theta_2 \in \mathscr{G}_2$ *and* $T \in \mathbf{A}$ *we have*

$$\theta_2(T^{\mathscr{G}_1}) = (\theta_2(T))^{\mathscr{G}_1}, \tag{3.2}$$

where $T \to T^{\mathscr{G}_1}$ *is the* $\mathscr{G}_1$*-canonical mapping of* $\mathbf{A}$.[15] *Denote by* $\mathscr{G}_{2,1}$ *the group of automorphisms of* $\mathbf{A}^{\mathscr{G}_1}$ *defined by* $\mathscr{G}_2$ *via* (3. 2). *Now if* $A^{\mathscr{G}_1}$ *is* $\mathscr{G}_{2,1}$*-finite then* $\mathbf{A}$ *is finite with respect to the group* $\mathscr{G} = \{\mathscr{G}_1, \mathscr{G}_2\}$ *generated by* $\mathscr{G}_1$ *and* $\mathscr{G}_2$. *Hence in this case* $\mathbf{A}$ *is* $\mathscr{G}_2$*-finite, too, and we have*

$$T^{\mathscr{G}} = (T^{\mathscr{G}_1})^{\mathscr{G}_2} = (T^{\mathscr{G}_2})^{\mathscr{G}_1} \qquad (T \in \mathbf{A}), \tag{3.3}$$

where $T \to T^{\mathscr{G}}$ *and* $T \to T^{\mathscr{G}_2}$ *are the corresponding* $\mathscr{G}$*- and* $\mathscr{G}_2$*-canonical mappings of* $\mathbf{A}$, *respectively.*

Proof. It is not hard to prove that $\mathbf{A}^{\mathscr{G}} = (\mathbf{A}^{\mathscr{G}_1})^{\mathscr{G}_{2,1}}$. Let now $\sigma \in \mathscr{R}^+(\mathbf{A}^{\mathscr{G}})$ be arbitrary. Since $\mathbf{A}^{\mathscr{G}_1}$ is $\mathscr{G}_{2,1}$-finite, in virtue of Corollary 1 of Proposition 3, σ can be extended to an element σ' of $\mathscr{R}^+(\mathbf{A}^{\mathscr{G}_1}, \mathscr{G}_{2,1})$. Since $\mathbf{A}$ is $\mathscr{G}_1$-finite, in virtue of the same corollary, σ' can be extended to an element σ'' of $\mathscr{R}^+(\mathbf{A}, \mathscr{G}_1)$. Now if $T \in \mathbf{A}$ and $\theta_2 \in \mathscr{G}_2$, then we have

$$\sigma''(\theta_2(T)) = \sigma''((\theta_2(T))^{\mathscr{G}_1}) = \sigma''(\theta_2(T^{\mathscr{G}_1})) = \sigma'(\theta_2(T^{\mathscr{G}_1})) = \\ = \sigma'(T^{\mathscr{G}_1}) = \sigma''(T^{\mathscr{G}_1}) = \sigma''(T),$$

that is $\sigma'' \in \mathscr{R}^+(\mathbf{A}, \mathscr{G})$. Hence, for every $T \in (\mathbf{A}^{\mathscr{G}})^+$, $T \neq O$ there exists an element σ of $\mathscr{R}^+(\mathbf{A}, \mathscr{G})$ such that $\sigma(T) \neq 0$, and this means that $\mathbf{A}$ is $\mathscr{G}$-finite. In particular, $\mathbf{A}$ is $\mathscr{G}_2$-finite, too. Now we are going to show that for every $T \in \mathbf{A}$

$$(T^{\mathscr{G}_1})^{\mathscr{G}_2} = (T^{\mathscr{G}_2})^{\mathscr{G}_1} \tag{3.4}$$

holds. Now let $T \in \mathbf{A}$ be arbitrary but fixed, and let $\{K_\iota(T)\}_{\iota \in I}$ be a net of elements of $\mathscr{K}_0(T, \mathscr{G}_2)$ such that

$$\lim_{\iota \in I}{}_{\mathrm{strong}}\, K_\iota(T) = T^{\mathscr{G}_2}. \tag{3.5}$$

Then

$$\lim_{\iota \in I}{}_{\mathrm{strong}}\, [K_\iota(T)]^{\mathscr{G}_1} = (T^{\mathscr{G}_2})^{\mathscr{G}_1}. \tag{3.6}$$

(cf. Theorem 2, (iv)). On the other hand, in virtue of (3. 2) we get that

$$[K_\iota(T)]^{\mathscr{G}_1} = K_\iota(T^{\mathscr{G}_1}). \tag{3.7}$$

Thus, in virtue of (3. 6) we have

$$\lim_{\iota \in I}{}_{\mathrm{strong}}\, K_\iota(T^{\mathscr{G}_1}) = (T^{\mathscr{G}_2})^{\mathscr{G}_1} \tag{3.8}$$

[15]) Condition (3. 2) is fulfilled for instance if every element of $\mathscr{G}_1$ commutes with every element of $\mathscr{G}_2$. In fact, to show this it is enough to take into account the construction of $T^{\mathscr{G}_1}$ and the continuity properties of the elements of $\mathscr{G}_2$.

This means that $(T^{\mathscr{G}_2})^{\mathscr{G}_1}$ belongs to $\mathscr{K}(T^{\mathscr{G}_1}, \mathscr{G}_2)$, and for every $\theta_2 \in \mathscr{G}_2$, we have $\theta_2((T^{\mathscr{G}_2})^{\mathscr{G}_1}) = (\theta_2(T^{\mathscr{G}_2}))^{\mathscr{G}_1} = (T^{\mathscr{G}_2})^{\mathscr{G}_1}$ (cf. (3. 2)) and this means that $(T^{\mathscr{G}_2})^{\mathscr{G}_1} \in \mathbf{A}^{\mathscr{G}_2} \cap \cap \mathscr{K}(T^{\mathscr{G}_1}, \mathscr{G}_2)$, that is

$$(T^{\mathscr{G}_2})^{\mathscr{G}_1} = (T^{\mathscr{G}_1})^{\mathscr{G}_2}.$$

Hence (3. 4) is proved. Now it is not hard to see that the mapping

$$T \to (T^{\mathscr{G}_1})^{\mathscr{G}_2} = (T^{\mathscr{G}_2})^{\mathscr{G}_1}$$

possesses all the properties of the mapping $T \to T^{\mathscr{G}}$. Thus, by the uniqueness part of Theorem 2, we get that

$$T^{\mathscr{G}} = (T^{\mathscr{G}_1})^{\mathscr{G}_2} = (T^{\mathscr{G}_2})^{\mathscr{G}_1},$$

which proves Proposition 4.

We think it is worth formulating Theorem 1 and Theorem 2 in the following well-known particular case (cf. [3], chap. III, § 4, Th. 3; § 5, Ex. 1).

Corollary to Theorems 1 and 2. *Let* $\mathbf{A}$ *be a finite von Neumann algebra, and denote by* $\mathbf{A}^{\natural}$ *its center. Then for every* $T \in \mathbf{A}$, *the set* $\mathbf{A}^{\natural} \cap \mathscr{K}(T, \mathscr{I}(\mathbf{A}))$ *consists of one element alone. Denote it by* $T^{\natural}$. *The mapping* $T \to T^{\natural}$ *has the following properties:*

(i) *for every* $T \in \mathbf{A}$ *and for every finite normal trace* ([3], chap. I, § 6, Def. 1) φ *on* $\mathbf{A}$ *we have* $\varphi(T^{\natural}) = \varphi(T)$,

(ii) $T \to T^{\natural}$ *is strictly positive and linear;*

(iii) $T \to T^{\natural}$ *is ultra-strongly and ultra-weakly continuous;*

(iv) *if* $T \in \mathbf{A}$ *and* U *is unitary in* $\mathbf{A}$ *then* $(U^*TU)^{\natural} = T^{\natural}$ *holds;*

(v) *if* $S \in \mathbf{A}^{\natural}$ *then* $S^{\natural} = S$;

(vi) *if* $S \in \mathbf{A}^{\natural}$ *and* $T \in \mathbf{A}$ *then* $(ST)^{\natural} = ST^{\natural}$.

Conversely, if there exists a positive normal linear mapping $T \to T'$ *of* $\mathbf{A}$ *onto* $\mathbf{A}^{\natural}$ *having properties analogous to* (iv) *and* (v), *then* $\mathbf{A}$ *is finite and* $T' = T^{\natural}$ *for every* $T \in \mathbf{A}$.

Proof. In Theorems 1 and 2 take $\mathscr{I}(\mathbf{A})$ for $\mathscr{G}$.

2. Let $\mathbf{A}$ be a von Neumann algebra in a Hilbert space $\mathfrak{H}$. Denote by $\mathbf{A}_U$ the group of all unitary elements of $\mathbf{A}$. Let $U \in \mathbf{A}_U$ be an arbitrary but fixed element of $\mathbf{A}_U$. For every $T \in \mathbf{L}(\mathfrak{H})$ [16]) put

$$T \to \theta_U(T) = U^*TU.$$

The set $\mathscr{G}(\mathbf{A}_U)$ of all possible θ_U is a group of automorphisms of $\mathbf{L}(\mathfrak{H})$. In the following we are going to characterize the von Neumann algebras $\mathbf{A}$ such that $\mathbf{L}(\mathfrak{H})$ is finite with respect to $\mathscr{G}(\mathbf{A}_U)$.

Proposition 5. *Let* $\mathbf{A}$ *be a von Neumann algebra in a Hilbert space* $\mathfrak{H}$. *Then* $\mathbf{L}(\mathfrak{H})$ *is* $\mathscr{G}(\mathbf{A}_U)$*-finite if and only if* $\mathbf{A}$ *is a product*[17]) *of finite discrete factors.*[18])

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[16]) $\mathbf{L}(\mathfrak{H})$ denotes the von Neumann algebra of all bounded linear operators of $\mathfrak{H}$.
[17]) Cf. [3], chap. I, § 2, no. 2.
[18]) Cf. [3], chap. I, § 8, no. 4.

Proof. Suppose that $\mathbf{A}$ is the product of the finite discrete factors $\mathbf{M}_\iota$ $(\iota \in I)$ that is

$$\mathbf{A} = \prod_{\iota \in I} \mathbf{M}_\iota .$$

It is evident that $(U_\iota)_{\iota \in I} \in \mathbf{A}_U$ if and only if $U_\iota \in (\mathbf{M}_\iota)_U$ [19]) for every $\iota \in I$. Furthermore, for every $\iota \in I$, the group $(\mathbf{M}_\iota)_U$ is compact in the weak operator topology. Thus, using the Tychonoff theorem on the topological product of compact spaces, it is not hard to see that $\mathbf{A}_U$ is compact in the weak topology. Denote by $\lambda(dU)$ the normalized Haar measure of $\mathbf{A}_U$, and let $T \in \mathbf{L}(\mathfrak{H})$ be arbitrary. If x is any element of $\mathfrak{H}$, the function

$$U \to f_{x,T}(U) = (U^* T U x | x)$$

is continuous on $\mathbf{A}_U$, since the weak and the strong topology coincide on $\mathbf{A}_U$. So

$$\int_{\mathbf{A}_U} f_{x,T}(U)\,\lambda(dU)$$

exists. Let $x \in \mathfrak{H}$ be fixed, and for every $T \in \mathbf{L}(\mathfrak{H})$ set

$$\sigma_x(T) = \int_{\mathbf{A}_U} f_{x,T}(U)\,\lambda(dU).$$

Using the unimodularity of λ and the properties of the integral, it is easy to show that $\sigma_x \in \mathscr{R}^+(\mathbf{L}(\mathfrak{H}), \mathscr{G}(\mathbf{A}_U))$. Now if $T \in \mathbf{L}^+(\mathfrak{H})$, $T \neq 0$ then there exists an element x_0 of $\mathfrak{H}$ such that $(Tx_0|x_0) > 0$. Then $\sigma_{x_0}(T) \neq O$, which proves that $\mathbf{L}(\mathfrak{H})$ is $\mathscr{G}(\mathbf{A}_U)$-finite.

Now suppose that $\mathbf{L}(\mathfrak{H})$ is $\mathscr{G}(\mathbf{A}_U)$-finite, and let $T \to T^{\mathscr{G}(\mathbf{A}_U)}$ be the $\mathscr{G}(\mathbf{A}_U)$-canonical mapping of $\mathbf{L}(\mathfrak{H})$ onto $\mathbf{L}(\mathfrak{H})^{\mathscr{G}(\mathbf{A}_U)}$ (cf. Theorem 2) which is equal to the commutant $\mathbf{A}'$ of $\mathbf{A}$. Let $\mathrm{Tr}(\cdot)$ be the canonical trace of $\mathbf{L}(\mathfrak{H})$ ([3], chap. I, § 6, no. 6), and let $S \in (\mathbf{A}')^+$, $S \neq O$ be arbitrary. Then there exists an element S_1 of $\mathbf{L}(\mathfrak{H})$ such that $O \leqq S_1 \leqq S$, $S_1 \neq O$, and $\mathrm{Tr}(S_1) < +\infty$. By the properties of the mapping $T \to T^{\mathscr{G}(\mathbf{A}_U)}$ we obtain that $O \leqq S_1^{\mathscr{G}(\mathbf{A}_U)} \leqq S^{\mathscr{G}(\mathbf{A}_U)} = S$. Furthermore, as $\mathrm{Tr}(\cdot)$ is lower semicontinuous in the weak topology ([3], chap. I, § 6, Prop. 2, Cor.) and $S_1^{\mathscr{G}(\mathbf{A}_U)} \in \mathscr{K}(S_1, \mathscr{G}(\mathbf{A}_U))$, we get that $\mathrm{Tr}(S_1^{\mathscr{G}(\mathbf{A}_U)}) \leqq \mathrm{Tr}(S_1)$. On the other hand, $S_1^{\mathscr{G}(\mathbf{A}_U)} \neq O$ since the mapping $T \to T^{\mathscr{G}(\mathbf{A}_U)}$ is strictly positive. So we have proved that for every $S \in (\mathbf{A}')^+$, $S \neq O$ there exists an element $S' \in (\mathbf{A}')^+$, $S' \neq O$, $S' \leqq S$ such that $\mathrm{Tr}(S') < +\infty$. Now let $E \neq O$ be a projection in $\mathbf{A}'$. Then there exists a non-zero element R of $(\mathbf{A}')^+$ with $R \leqq E$ and $\mathrm{Tr}(R) < +\infty$. Let $R = \int \lambda\, dF_\lambda$ be the spectral representation of R and set $F = I - F_{\frac{\|R\|}{2}+0}$. Then it is evident that $F \in \mathbf{A}'$, $F \neq O$ and $\frac{\|R\|}{2} F \leqq R$. Therefore, $\mathrm{Tr}(F) < +\infty$. Furthermore, as F is a projection, we obtain that $F \leqq E$. Let now F_0 be any of the projections of $\mathbf{A}'$ such that $F_0 \neq O$, $F_0 \leqq E$ and $\mathrm{Tr}(F_0)$ is minimal. Then F_0 is minimal in $\mathbf{A}'$. Indeed, $F_0' \in \mathbf{A}'$, $F_0' \neq O$, $F_0' \neq F_0$, $F_0' \leqq F_0$ would imply $F_0' \leqq E$, $\mathrm{Tr}(F_0') < +\infty$ and $\mathrm{Tr}(F_0') < \mathrm{Tr}(F_0)$ which contradicts the minimality of $\mathrm{Tr}(F_0)$. Thus, every non-zero projection of $\mathbf{A}'$ majorizes a non-zero minimal projection of $\mathbf{A}'$. Hence, in virtue

[19]) $(\mathbf{M}_\iota)_U$ denotes the group of the unitary elements of $\mathbf{M}_\iota$.

of Ex. 4, p. 126 of [3], $\mathbf{A}'$ and so $\mathbf{A}$ is a product of discrete factors. Since $\mathbf{A}$ is finite, each factor occuring in the decomposition of $\mathbf{A}$ is finite ([3], chap. I, § 8, no. 2). Thus the proof of Proposition 5 is comlete.

Corollary. *In order that the group $\mathbf{A}_U$ of the unitary elements of a von Neumann algebra $\mathbf{A}$ be compact in the weak topology, it is necessary and sufficient that $\mathbf{A}$ be the product of finite discrete factors.*

Proof. The sufficiency of our condition is evident by the Tychonoff theorem (cf. the first step of the proof of Proposition 5). Now, if $\mathbf{A}_U$ is weakly compact, then arguing in the same way as in the proof of Proposition 5, we obtain that $\mathbf{L}(\mathfrak{H})$ is $\mathscr{G}(\mathbf{A}_U)$-finite which means, by Proposition 5, that $\mathbf{A}$ is a product of finite discrete factors. Hence the proof of Corollary is complete.

Bibliography

[1] J. Dixmier, Les anneaux d'opérateurs de classe finie, *Ann. Ec. Norm. Sup.*, **66** (1949), 209—261.
[2] J. Dixmier, Formes linéaires sur un anneau d'opérateurs, *Bull. Soc. Math. Fr.*, **81** (1953), 9—39.
[3] J. Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien (Algèbres de von Neumann)* (Paris, 1957).
[4] K. Jacobs, *Neuere Methoden und Ergebnisse der Ergodentheorie* (Berlin—Göttingen—Heidelberg, 1960).
[5] I. Kovács et J. Szűcs, Théorèmes de type ergodique dans les algèbres de von Neumann, *C. R. Acad. Sci. Paris*, **262** (1966), 341—344.
[6] S. Sakai, On topological properties of W^*-algebras, *Proc. Japan Acad.*, **33** (1957), 439—444.
[7] J. Schwartz, Two finite, non-hyperfinite, non-isomorphic factors, *Comm. Pure Appl. Math.*, **16** (1963), 19—26.

(Received June 16, 1966)

On unitary ϱ-dilations of operators

By E. DURSZT in Szeged

Using the notation of [1], $\mathscr{C}_\varrho$ ($\varrho \geqq 0$) will denote the class of those (bounded, linear) operators T in a Hilbert space $\mathfrak{H}$, for which

$$T^n h = \varrho P U^n h \quad (h \in \mathfrak{H}; n = 1, 2, \ldots)$$

holds, where U is a unitary operator in some Hilbert space $\mathfrak{K}$, containing $\mathfrak{H}$ as a subspace, and P denotes the projection of $\mathfrak{H}$ onto $\mathfrak{K}$.

The following theorem was proved in [1].

Theorem A. *In the case* $\varrho > 0$, $T \in \mathscr{C}_\varrho$ *if and only if*

$$(\mathrm{I}_\varrho) \qquad \|h\|^2 - 2\left(1 - \frac{1}{\varrho}\right) \mathrm{Re}\,(zTh, h) + \left(1 - \frac{2}{\varrho}\right) \|zTh\|^2 \geqq 0 \text{ for } h \in \mathfrak{H}, |z| \leqq 1;$$

(II) *the spectrum of* T *lies in the closed unit disc.*

The purpose of this paper is to study the monotonity properties of $\mathscr{C}_\varrho$ as a function of ϱ. Meanwhile we shall give a simple necessary and sufficient condition for a normal T to belong to $\mathscr{C}_\varrho$.

We start with the following

Lemma 1. $\mathscr{C}_\varrho$ *is a non-decreasing function of* ϱ *in the sense that* $\mathscr{C}_{\varrho_1} \subset \mathscr{C}_{\varrho_2}$ *if* $0 \leqq \varrho_1 < \varrho_2$.

This lemma was already proved in [1]. Here we give another proof of it as follows.

Proof. The definition of $\mathscr{C}_0$ shows that $T \in \mathscr{C}_0$ if and only if $T = O$ (the zero operator). According to Theorem A we have $O \in \mathscr{C}_\varrho$ for every $\varrho > 0$. This implies our lemma in the case $\varrho_1 = 0$.

Now let $\varrho_1 > 0$, and set

$$F_{z,h}(\varrho) = \varrho \|h\|^2 - 2(\varrho - 1)\,\mathrm{Re}\,(z\,Th, h) + (\varrho - 2)\|zTh\|^2 =$$
$$= \|h\|^2 - \|zTh\|^2 + (\varrho - 1)\|(I - zT)h\|^2 \quad (\varrho > 0, |z| \leqq 1, h \in \mathfrak{H}).$$

(I_ϱ) holds if and only if $F_{z,h}(\varrho) \geqq 0$ whenever $|z| \leqq 1$ and $h \in \mathfrak{H}$. Now let $T \in \mathscr{C}_{\varrho_1}$ and $\varrho_2 > \varrho_1$. In this case (II) holds, and $F_{z,h}(\varrho_1) \geqq 0$ ($|z| \leqq 1, h \in \mathfrak{H}$). $F_{z,h}(\varrho)$ is a monotone non-decreasing function of ϱ, consequently we also have $F_{z,h}(\varrho_2) \geqq 0$. This implies $T \in \mathscr{C}_{\varrho_2}$.

Theorem 1. *In the case that T is normal, a necessary and sufficient condition for $T\in\mathscr{C}_\varrho$ is*

$$\|T\| \leqq \begin{cases} \dfrac{\varrho}{2-\varrho}, & \text{if } 0\leqq\varrho\leqq 1, \\ 1, & \text{if } \varrho>1. \end{cases}$$

Proof. In the case that $\varrho=0$ our statement is trivial.

Let $0<\varrho<2$. In this case (I_ϱ) is equivalent with

$$\frac{\varrho}{\varrho-2}\|h\|^2-2\frac{\varrho-1}{\varrho-2}\operatorname{Re}(zTh,h)+\|zTh\|^2\leqq 0 \qquad (h\in\mathfrak{H},\quad |z|\leqq 1).$$

This latter relation holds if and only if

$$\left\|\left(\frac{\varrho-1}{\varrho-2}I-zT\right)h\right\|^2+\frac{\varrho}{\varrho-2}\|h\|^2\leqq\left(\frac{\varrho-1}{\varrho-2}\right)^2\|h\|^2 \qquad (h\in\mathfrak{H},\quad |z|\leqq 1)$$

or, equivalently,

$$\left\|\left(\frac{1-\varrho}{2-\varrho}I-zT\right)h\right\|\leqq\frac{1}{2-\varrho}\|h\| \qquad (h\in\mathfrak{H},\quad |z|\leqq 1),$$

i. e.

$$(\mathrm{I}'_\varrho) \qquad \sup_{|z|\leqq 1}\left\|\frac{1-\varrho}{2-\varrho}I+zT\right\|\leqq\frac{1}{2-\varrho}.$$

So we see that

(1) (I_ϱ) *and* (I'_ϱ) *are equivalent for* $0<\varrho<2$.

Moreover,

$$\left\|\frac{1-\varrho}{2-\varrho}I+zT\right\|\leqq\left\|\frac{1-\varrho}{2-\varrho}I\right\|+\|zT\|,$$

consequently

$$(2) \qquad \sup_{|z|\leqq 1}\left\|\frac{1-\varrho}{2-\varrho}I+zT\right\|\leqq\frac{1-\varrho}{2-\varrho}+\|T\|.$$

Now let T be normal. Then the spectrum of T contains a complex number of modulus $\|T\|$, say $\zeta^{-1}\|T\|$ and this is an approximative proper value of T, i.e. for every $\varepsilon>0$ there exists an element h_ε of $\mathfrak{H}$ such that $\|h_\varepsilon\|=1$ and

$$\|(\|T\|I-\zeta T)h_\varepsilon\|<\varepsilon.$$

Using this fact we have

$$\left\|\left(\frac{1-\varrho}{2-\varrho}I+\zeta T\right)h_\varepsilon\right\|\geqq\left\|\left(\frac{1-\varrho}{2-\varrho}I+\|T\|I\right)h_\varepsilon\right\|-\|(\|T\|I-\zeta T)h_\varepsilon\|>\frac{1-\varrho}{2-\varrho}+\|T\|-\varepsilon.$$

This is true for every $\varepsilon>0$, consequently

$$\sup_{|z|\leqq 1}\left\|\frac{1-\varrho}{2-\varrho}I+zT\right\|\geqq\frac{1-\varrho}{2-\varrho}+\|T\|.$$

Our latter relation and (2) show that

$$\sup_{|z|\leq 1}\left\|\frac{1-\varrho}{2-\varrho}I+zT\right\|=\frac{1-\varrho}{2-\varrho}+\|T\|.$$

This implies that, in the case that T is normal, (I'_ϱ) is equivalent to

$$\frac{1-\varrho}{2-\varrho}+\|T\|\leqq\frac{1}{2-\varrho}$$

or to

(I''_ϱ) $$\|T\|\leqq\frac{\varrho}{2-\varrho}.$$

Using (1) we have:

(I_ϱ) *and* (I''_ϱ) *are equivalent for* $0<\varrho<2$ *if* T *is normal.*

Now let $0<\varrho\leqq 1$. In this case (I''_ϱ) implies (II). Moreover, using Theorem A, we have: In the case $0<\varrho\leqq 1$ and T is normal, $T\in\mathscr{C}_\varrho$ if and only if (I''_ϱ) holds.

Now, for normal T, (II) is equivalent to the condition $\|T\|\leqq 1$. Thus, by Lemma 1, if T is normal, we have $T\in\mathscr{C}_\varrho$ for $\varrho>1$ if and only if $\|T\|\leqq 1$.

So we finished the proof.

For $0\leqq\varrho\leqq 1$, $\dfrac{\varrho}{2-\varrho}$ is strictly increasing function of ϱ. Thus, by Theorem 1 and Lemma 1, $\mathscr{C}_\varrho$ is a strictly increasing function of ϱ for $0\leqq\varrho\leqq 1$.

If $\dim\mathfrak{H}=1$ then there exist only operators of multiplication by complex numbers, and these are normal. In this case, Theorem 1 shows the monotonity properties of $\mathscr{C}_\varrho$.

Theorem 2. *If* $\dim\mathfrak{H}\geqq 2$ *then* $\mathscr{C}_\varrho$ *is a strictly increasing function of* ϱ *in the sense that*

$$\mathscr{C}_{\varrho_1}\subset\mathscr{C}_{\varrho_2}\quad\text{and}\quad\mathscr{C}_{\varrho_1}\neq\mathscr{C}_{\varrho_2}\quad\text{if}\quad 0\leqq\varrho_1<\varrho_2.$$

Proof. For arbitrary $\varrho\geqq 0$ we shall construct an operator T_ϱ such that $T_\varrho\in\mathscr{C}_\varrho$ and $\|T_\varrho\|=\varrho$. This T_ϱ does not belong to $\mathscr{C}_\sigma$ if $0\leqq\sigma<\varrho$. This fact and Lemma 1 will prove our theorem.

Let

(4) $$\{\varphi_1,\varphi_2,\psi_\nu\ (\nu\in\Omega)\}$$

be an orthonormal basis in $\mathfrak{H}$. We define T_ϱ by

(5) $$T_\varrho\varphi_1=\varrho\varphi_2,\quad T_\varrho\varphi_2=0,\quad T_\varrho\psi_\nu=0\qquad(\nu\in\Omega).$$

Evidently, $\|T_\varrho\|=\varrho$ and

(6) $$T_\varrho^n=O\qquad(n=2,3,\ldots).$$

Let us construct an orthonormal system

(7) $$\{\varphi'_m\ (m=0,\pm 1,\ldots),\ \psi'_{\nu,m}\ (\nu\in\Omega,\ m=0,\pm 1,\ldots)\}$$

and identify φ_k with φ'_k $(k=1,2)$ and ψ_ν with $\psi'_{\nu,0}$. So the Hilbert space $\mathfrak{K}$ spanned by the system (7) will contain $\mathfrak{H}$ as a subspace. Let P be the orthogonal projection of $\mathfrak{R}$ onto $\mathfrak{H}$ and define the linear operator U on $\mathfrak{R}$ by

$$U\varphi'_m = \varphi'_{m+1}, \quad U\psi'_{\nu,m} = \psi'_{\nu,m+1} \qquad (\nu \in \Omega; \quad m=0, \pm 1, \ldots).$$

Evidently, U is unitary and we have

$$\varrho PU\varphi_1 = \varrho P\varphi_2 = \varrho\varphi_2, \quad \varrho PU\varphi_2 = \varrho P\varphi'_3 = 0, \quad \varrho PU\psi_\nu = \varrho P\psi'_{\nu,1} = 0.$$

Consequently, by (5),

$$Th = \varrho PUh \qquad (h \in \mathfrak{H}). \tag{8}$$

For $m \geqq 2$ we get

$$\varrho PU^m \varphi_k = \varrho P\varphi'_{k+m} = 0, \quad \varrho PU^m \psi_\nu = \varrho P\psi'_{\nu,m} = 0 \qquad (k=1,2; \quad \nu \in \Omega),$$

consequently,

$$T^n h = 0 \qquad (h \in \mathfrak{H}, \quad n \geqq 2).$$

Thus, by (6) and (8), $T_\varrho \in \mathscr{C}_\varrho$.

So the proof is complete.

Reference

[1] B. Sz.-Nagy and C. Foiaş, On certain classes of power-bounded operators in Hilbert space, *Acta Sci. Math.*, **27** (1966), 17—25.

(Received June 1, 1966)

Cellularity of a suspension arc

By P. H. DOYLE in East Lansing (Michigan, U. S. A.)

Let X be a topological space that is locally euclidean of dimension n except at a single point p. We assume that the suspension $S(X)$ of X is an $(n+1)$-sphere, S^{n+1}. That there are many such spaces X follows from [3]. If v_1 and v_2 are the suspension points of $S(X)$ we study the embedding of $S(p)=v_1p\cup v_2p$, the suspension arc, in S^{n+1}.

Lemma 0. *The arc v_1p in S^{n+1} is cellular.*

Proof. Consider $S(X)-v_1p$. This set is the manifold with boundary v_2X-p with an open collar attached to its boundary. Thus it is topologically E^{n+1} and v_1p is therefore cellular.

Since the above argument applies equally well to v_2p, $S(p)$ is the union of two cellular arcs meeting in a common endpoint p. In general such an arc is not cellular. Example 1.1 of [2] gives such an arc in S^4. If we call this arc A in S^3 and take X to be S^3 modulo A, then $S(X)$ is S^4. However the suspension arc in this case is not cellular since its fundamental group is the non-trivial group $\pi_1(S^3-A)$. We give a sufficient condition that the suspension arc be cellular that is purely geometric.

Theorem. *If there is an n-cell C^n in $S^{n+1}-v_2$ such that v_1 lies in the interior of C^n while the boundary of C^n meets $S(p)$ in just one point, then $S(p)$ is cellular.*

Proof. By the hypothesis it is clear that v_2 does not lie in C^n. Since S^{n+1} is a suspension then given any open set U in S^{n+1} containing v_1 there is a homeomorphism h of S^{n+1} onto itself that carries $S(p)$ onto itself and $h(C^n)\subset U$. But then by Lemma 1 of [1] and Lemma 0 above $S(p)$ is cellular in S^{n+1}.

References

[1] P. H. DOYLE, A sufficient condition that an arc in S^n be cellular, *Pacific J. Math.*, **14** (1964), 501—503.

[2] R. H. FOX, and E. ARTIN, Some wild cells and spheres in three-dimensional space, *Ann. Math.*, **49** (1948), 979—990.

[3] K. W. KWUN, Product of euclidean spaces modulo an arc, *Ann. Math.*, **79** (1964), 104—108.

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(Received April 22, 1966)

A convergence theorem of orthogonal series

By P. RÉVÉSZ in Budapest

Introduction

Let $\varphi_1(x), \varphi_2(x), \dots$ be an orthonormal sequence defined on a measure space $\{X, S, \mu\}$. For the sake of simplicity we assume that $\mu(X)=1$. Further let $c_1, c_2, \dots$ be a sequence of real numbers with

$$\sum_{i=1}^{\infty} c_i^2 < \infty. \tag{1}$$

A fundamental problem of the theory of orthogonal series is to find conditions implying the almost everywhere convergence of the series

$$\sum_{i=1}^{\infty} c_i \varphi_i(x). \tag{2}$$

In general the condition (1) does not imply the almost everywhere convergence of the series (2). However, the classical Menšov—Rademacher theorem states:

Theorem A. *If*

$$\sum_{i=1}^{\infty} c_i^2 \log^2 i < \infty \tag{3}$$

then the series (2) *is convergent almost everywhere.*

Under certain special restrictions on the sequence $\{\varphi_k(x)\}$ the condition (3) can be replaced by weaker ones. For example the classical Kolmogorov theorem states that *if the functions $\varphi_1(x), \varphi_2(x), \dots$ are independent in the sense of probability theory, with expectation* 0 *and variance* 1, *then condition* (1) *implies the almost everywhere convergence of* (2). A similar result is due to G. ALEXITS [1]. He introduced the following definitions:

Definition 1. The sequence $\varphi_1(x), \varphi_2(x), \dots$ of measurable functions is called a multiplicative system if

$$\int_X \varphi_{i_1} \varphi_{i_2} \dots \varphi_{i_k} \, d\mu = 0 \qquad (i_1 < i_2 < \dots < i_k; \quad k = 1, 2, \dots).$$

Definition 2. The sequence $\varphi_1(x), \varphi_2(x), \dots$ of measurable functions is called a strongly multiplicative system if the system $\{\varphi_{i_1}\varphi_{i_2} \dots \varphi_{i_k}\}$ is an orthogonal system, i.e. if

$$\int_X \varphi_{i_1}^{\alpha_1} \varphi_{i_2}^{\alpha_2} \dots \varphi_{i_k}^{\alpha_k}\, d\mu = 0 \qquad (i_1 < i_2 < \dots < i_k; \quad k = 1, 2, \dots)$$

where $\alpha_1, \alpha_2, \dots, \alpha_k$ can be equal to 1 or 2 but at least one element of the sequence $\alpha_1, \alpha_2, \dots, \alpha_k$ is equal to 1.

Definition 3. The sequence $\varphi_1(x), \varphi_2(x), \dots$ of measurable functions is called an equinormed strongly multiplicative system (ESMS) if

$$\int_X \varphi_i\, d\mu = 0, \quad \int_X \varphi_i^2\, d\mu = 1 \qquad (i = 1, 2, \dots),$$

(4)

$$\int_X \varphi_{i_1}^{\alpha_1} \varphi_{i_2}^{\alpha_2} \dots \varphi_{i_k}^{\alpha_k}\, d\mu = \int_X \varphi_{i_1}^{\alpha_1}\, d\mu \int_X \varphi_{i_2}^{\alpha_2}\, d\mu \dots \int_X \varphi_{i_k}^{\alpha_k}\, d\mu \quad (i_1 < \dots < i_k; \quad k = 1, 2, \dots)$$

where $\alpha_1, \alpha_2, \dots, \alpha_k$ can be equal to 1 or 2.

Making use of these definitions, ALEXITS and TANDORI ([2]) proved the following

Theorem B. *If $\varphi_1(x), \varphi_2(x), \dots$ is a uniformly bounded ESMS, then condition* (1) *implies the almost everywhere convergence of the series* (2).

Obviously any independent system with $\int_X \varphi_i d\mu = 0$, $\int_X \varphi_i^2 d\mu = 1$ is an ESMS, therefore ALEXITS's theorem would be much stronger the KOLMOGOROV's if the condition of boundedness could be dropped. A previous paper ([3]) of the author shows that there are some further theorems (the central limit theorem and the law of the iterated logarithm) known to hold for independent random variables which remain valid for ESMS.

§ 1. The Theorem

The aim of this paper is to prove the following

Theorem 1. *Let $\varphi_1(x), \varphi_2(x), \dots$ be a sequence of measurable functions defined on a measure space $\{X, S, \mu\}$ with $\mu(X) = 1$. Suppose that*

(5)
$$\int_X \varphi_i^4\, d\mu \leqq K \qquad (i = 1, 2, \dots)$$

and

(6)
$$\int_X \varphi_i^2 \varphi_j \varphi_k\, d\mu = \int_X \varphi_i^2 \varphi_j\, d\mu = \int_X \varphi_i \varphi_j \varphi_k \varphi_l\, d\mu = \\ = \int_X \varphi_i \varphi_j \varphi_k\, d\mu = \int_X \varphi_i \varphi_j\, d\mu = \int_X \varphi_i\, d\mu = 0,$$

where the indices i, j, k, l are different, and K is a positive constant. Further let $c_1, c_2, \dots$ be a sequence of real numbers and suppose that there exists an integer r (depending on $\{c_k\}$) such that

$$\sum_{k=1}^{\infty} c_k^2 l_r^2(k) < \infty \tag{7}$$

where[1]

$$l(x) = l_1(x) = \begin{cases} \log x & \text{if} \quad x \geqq 2 \\ 1 & \text{if} \quad 0 < x < 2 \end{cases}$$

and $l_r(x)$ is the r-th iterate of $l(x)$ i.e. $l_r(x) = l(l_{r-1}(x))$. Then the series

$$\sum_{k=1}^{\infty} c_k \varphi_k(x)$$

is convergent almost everywhere.

Remark 1. If $\{\varphi_k\}$ is a sequence of fourwise independent random variables with expectation 0 and variance 1 and with uniformly bounded fourth moments then (5) and (6) hold.

Remark 2. Condition (7) is not very far from condition (1). This facts suggests the conjecture that (7) can be replaced by (1).

The proof of this theorem is based on three lemmas.

Lemma 1. *If $\varphi_1, \varphi_2, \dots$ is a sequence of measurable functions for which (5) and (6) hold, then*

$$\int_X \max_{1 \leqq k \leqq n} |c_1 \varphi_1 + c_2 \varphi_2 + \dots + c_k \varphi_k|^4 \, d\mu \leqq 8Kl^4(n) \left(\sum_{j=1}^{n} c_j^2 \right)^2 \tag{8}$$

where $c_1, c_2, \dots, c_n$ is an arbitrary sequence of real numbers.

Remark 3. This lemma is not the best possible. In [3] it was proved that in the case $c_1 = c_2 = \dots = c_n = 1$, $l^4(n)$ can be replaced by $O(1) l^3(n)$. The same method can be applied in this more general case to obtain a stronger inequality. Unfortunately using such a stronger inequality instead of (8) we cannot obtain a stronger result than Theorem 1, therefore we do not intend to attain the best possible inequality.

Lemma 2. *If $c_1, c_2, \dots$ is a sequence of real numbers for which*

$$\sum_{k=1}^{\infty} c_k^2 l_r^2(k) < \infty$$

then there exists a sequence $n_1, n_2, \dots$ of integers for which

$$\sum_{k=1}^{\infty} \left(\sum_{j=n_k+1}^{n_{k+1}} c_j^2 \right) l_{r-1}^2(k) < \infty, \tag{9}$$

$$\sum_{k=1}^{\infty} \left(\sum_{j=n_k+1}^{n_{k+1}} c_j^2 \right)^2 l^4(n_{k+1} - n_k) < \infty. \tag{10}$$

[1]) $\log x$ means the logarithm with the base 2.

Lemma 3. *If $\{\varphi_k\}$ is a sequence of measurable functions for which* (5) *and* (6) *hold and $m_1 < m_2 < \dots$ is a sequence of integers then for the sequence*

$$\psi_k = \begin{cases} \dfrac{1}{\alpha_k} \sum\limits_{j=m_k+1}^{m_{k+1}} c_j \varphi_j & \text{if } \alpha_k > 0 \\ 0 & \text{if } \alpha_k = 0 \end{cases} \tag{11}$$

where

$$\alpha_k = \left[\sum_{j=m_k+1}^{m_{k+1}} c_j^2 \right]^{1/2},$$

we have

$$\int_X \psi_k^4 \, d\mu \leqq 4K$$

and (6).

§ 2. The proof

Proof of Lemma 1. First of all we assume that $n = 2^\nu$ ($\nu = 1, 2, \dots$) and introduce the following notations

$$\sigma_j = c_1\varphi_1 + c_2\varphi_2 + \dots + c_j\varphi_j \qquad (j = 1, 2, \dots),$$
$$\psi_{\alpha\beta} = c_{\alpha+1}\varphi_{\alpha+1} + c_{\alpha+2}\varphi_{\alpha+2} + \dots + c_\beta\varphi_\beta$$

where $\alpha = \mu 2^k$; $\beta = (\mu+1)2^k$; $\mu = 0, 1, 2, \dots, 2^{\nu-k} - 1$; $k = 0, 1, 2, \dots, \nu - 1$. Consider the function σ_j as the sum of some $\psi_{\alpha\beta}$. Let us put

$$\sigma_j = \sum_i \psi_{\alpha_i \beta_i}$$

where $\beta_1 - \alpha_1 > \beta_2 - \alpha_2 > \dots$. Clearly the number of the members of the sum $\sum\limits_i \psi_{\alpha_i\beta_i}$ is less than ν. Therefore by the Schwarz inequality we have

$$\sigma_j^4 = \left(\sum_i \psi_{\alpha_i \beta_i} \right)^4 \leqq \nu^2 \left(\sum_i \psi_{\alpha_i \beta_i}^2 \right)^2 \leqq \nu^3 \sum_i \psi_{\alpha_i \beta_i}^4$$

which implies

$$\int_X \max_{1 \leqq j \leqq 2^\nu} \sigma_j^4 \, d\mu \leqq \nu^3 \sum_{\alpha, \beta} \int_X \psi_{\alpha\beta}^4 \, d\mu \tag{12}$$

where α and β run over all their possible values.

We obtain an estimation of the right hand side of (12) as follows

$$\begin{aligned} \int_X \psi_{\alpha\beta}^4 \, d\mu = \sum_{j=\alpha+1}^{\beta} c_j^4 \int_X \varphi_j^4 \, d\mu + 6 \sum_{\alpha < i < j \leqq \beta} c_i^2 c_j^2 \int_X \varphi_i^2 \varphi_j^2 \, d\mu + \\ + 4 \sum_{\substack{\alpha < i, j \leqq \beta \\ i \neq j}} c_i^3 c_j \int_X \varphi_i^3 \varphi_j \, d\mu \leqq K \left\{ \sum_{j=\alpha+1}^{\beta} c_j^4 + 6 \sum_{\alpha < i < j \leqq \beta} c_i^2 c_j^2 + 4 \sum_{\substack{\alpha < i, j \leqq \beta \\ i \neq j}} |c_i^3 c_j| \right\}. \end{aligned} \tag{13}$$

Summing the right hand side of (13) for each α, β we obtain any member of it at most ν times, so we have

$$\sum_{\alpha,\beta}\int_X \psi_{\alpha\beta}^4\,d\mu \leqq \nu K\left\{\sum_{j=1}^{2^\nu} c_j^4 + 6\sum_{1\leqq i<j\leqq 2^\nu} c_i^2 c_j^2 + 4\sum_{\substack{1\leqq i,j\leqq 2^\nu\\ i\neq j}} |c_i^3 c_j|\right\}.$$

It is easy to see that

$$\sum_{\substack{1\leqq i,j\leqq 2^\nu\\ i\neq j}} |c_i^3 c_j| \leqq \frac{1}{2}\sum_{j=1}^{2^\nu} c_j^4 + \sum_{1\leqq i<j\leqq 2^\nu} c_i^2 c_j^2.$$

Hence we have

$$\sum_{\alpha,\beta}\int_X \psi_{\alpha\beta}^4\,d\mu \leqq 4\nu K\left(\sum_{j=1}^{2^\nu} c_j^2\right)^2.$$

Thus in the case $n=2^\nu$ we obtained

$$\int_X \sup_{1\leqq k\leqq n} |c_1\varphi_1 + c_2\varphi_2 + \dots + c_k\varphi_k|^4\,d\mu \leqq 4Kl^4(n)\left(\sum_{j=1}^{n} c_j^2\right)^2.$$

Our inequality in the case $2^\nu \leqq n < 2^{\nu+1}$ follows immediately from this fact, setting

$$c_{n+1} = c_{n+2} = \dots = c_{2^{\nu+1}} = 0$$

and using the inequality

$$2\log^4 n \geqq (\log 2n)^4 \geqq (\nu+1)^4$$

if n is large enough.

Proof of Lemma 2. Set

$$A = \sum_{k=1}^{\infty} c_k^2 l_r^2(k)$$

then

$$A \geqq \sum_{k=n}^{\infty} c_k^2 l_r^2(k) \geqq l_r^2(n)\sum_{k=n}^{\infty} c_k^2$$

and

$$\sum_{k=n}^{\infty} c_k^2 \leqq \frac{A}{l_r^2(n)}.$$

Therefore we have

$$\sum_{k=2^\nu+1}^{2^{\nu+1}} c_k^2 \leqq \frac{A}{l_{r-1}^2(\nu)}. \tag{14}$$

Now we can find between $2^\nu+1$ and $2^{\nu+1}$ a sequence of integers

$$2^\nu + 1 = \tau_0^{(\nu)} \leqq \tau_1^{(\nu)} \leqq \dots \leqq \tau_{s_\nu-1}^{(\nu)} \leqq \tau_{s_\nu}^{(\nu)} = 2^{\nu+1}$$

as follows: Let $\tau_2^{(\nu)}$ be the smallest integer for which

$$\sum_{j=2^\nu+1}^{\tau_2^{(\nu)}} c_j^2 \geqq \frac{A}{\nu^6 l_{r-1}^2(\nu)}$$

and let $\tau_1^{(\nu)}=\tau_2^{(\nu)}-1$. Similarly let $\tau_4^{(\nu)}$ be the smallest integer for which

$$\sum_{j=\tau_2^{(\nu)}+1}^{\tau_4^{(\nu)}} c_j^2 \geqq \frac{A}{\nu^6 l_{r-1}^2(\nu)}$$

and let $\tau_3^{(\nu)}=\tau_4^{(\nu)}-1$. In general, if $\tau_{2l}^{(\nu)}$ is defined, we define $\tau_{2(l+1)}^{(\nu)}$ as the smallest integer for which

$$\sum_{j=\tau_{2l}^{(\nu)}+1}^{\tau_{2(l+1)}^{(\nu)}} c_j^2 \geqq \frac{A}{\nu^6 l_{r-1}^2(\nu)}$$

and let $\tau_{2l+1}^{(\nu)}=\tau_{2(l+1)}^{(\nu)}-1$. Now let

$$2^\nu+1 = t_0^{(\nu)}<t_1^{(\nu)}<\ldots<t_{p_\nu}^{(\nu)} = 2^{\nu+1}$$

be the different elements of the sequence $\tau_0^{(\nu)}, \tau_1^{(\nu)}, \ldots, \tau_{s_\nu}^{(\nu)}$. Clearly

$$p_\nu \leqq 2\nu^6.$$

Define now the sequence $\{n_k\}$ as the union of the sequences $t_0^{(\nu)}, t_1^{(\nu)}, \ldots, t_{p_\nu}^{(\nu)}$ i.e. the sequence $n_1, n_2, \ldots$ is the same as the sequence

$$t_0^{(1)}, t_1^{(1)}, t_0^{(2)}, t_1^{(2)}, t_2^{(2)}, t_0^{(3)}, t_1^{(3)}, \ldots, t_{p_3}^{(3)}, t_0^{(4)}, t_1^{(4)}, \ldots, t_{p_4}^{(4)}, t_0^{(5)}, t_1^{(5)}, \ldots, t_{p_5}^{(5)}, \ldots .$$

Clearly if $n_k\in(2^\nu, 2^{\nu+1}]$ then $k\leqq 2\sum_{j=1}^{\nu} j^6 \leqq 2(\nu+1)^7$. We prove that (9) and (10) hold for this sequence $\{n_k\}$. We have

$$A = \sum_{j=1}^{\infty} c_j^2 l_r^2(j) = \sum_{\nu=1}^{\infty} \sum_{j=2^\nu+1}^{2^{\nu+1}} c_j^2 l_r^2(j) \geqq \frac{1}{2} \sum_{\nu=1}^{\infty} \sum_{j=2^\nu+1}^{2^{\nu+1}} c_j^2 l_{r-1}^2(\nu+1) =$$

$$= \frac{1}{2} \sum_{\nu=1}^{\infty} \sum_{n_k\in(2^\nu, 2^{\nu+1}]} \sum_{j=n_k+1}^{n_{k+1}} c_j^2 l_{r-1}^2(\nu+1) \geqq$$

$$\geqq \frac{1}{4} \sum_{\nu=1}^{\infty} \sum_{n_k\in(2^\nu, 2^{\nu+1}]} \sum_{j=n_k+1}^{n_{k+1}} c_j^2 l_{r-1}^2\big(2(\nu+1)^7\big) \geqq \frac{1}{4} \sum_{k=1}^{\infty} \sum_{j=n_k+1}^{n_{k+1}} c_j^2 l_{r-1}^2(k)$$

that proves (9).

If $n_k\in(2^\nu, 2^{\nu+1}]$ then by the definition of $\{n_k\}$ we have

$$n_{k+1}-n_k \leqq 2^{\nu+1}$$

and either

$$\sum_{j=n_k+1}^{n_{k+1}} c_j^2 \leqq \frac{A}{\nu^6 l_{r-1}^2(\nu)}$$

or

$$n_{k+1}-n_k = 1;$$

this gives (10).

Proof of Lemma 3 is so simple that we can omit it.

Proof of Theorem 1. First of all we prove that the series (2) is convergent almost everywhere if

$$\sum_{k=1}^{\infty} c_k^2\, l_2^2(k) < \infty. \tag{15}$$

Let $\{n_k\}$ be a sequence of integers for which

$$\sum_{k=1}^{\infty} \left(\sum_{j=n_k+1}^{n_{k+1}} c_j^2 \right) l^2(k) < \infty \tag{16}$$

and (10) holds. Set

$$\psi_k = \begin{cases} \dfrac{1}{\alpha_k} \displaystyle\sum_{j=n_k+1}^{n_{k+1}} c_j \varphi_j & \text{if} \quad \alpha_k > 0 \\ 0 & \text{if} \quad \alpha_k = 0 \end{cases}$$

where $\alpha_k = \left[\sum_{j=n_k+1}^{n_{k+1}} c_j^2 \right]^{1/2}$ and put $\sigma_N = \sum_{j=1}^{N} c_j \varphi_j$. Clearly we have

$$\sigma_{n_K} = \sum_{k=1}^{K-1} \alpha_k \psi_k. \tag{17}$$

By Lemma 3, Theorem A and (16), the sequence $\{\sigma_{n_K}\}$ is convergent almost everywhere. By Lemma 1 and (10)

$$\sum_{k=1}^{\infty} \int_X \max_{n_k < j \leqq n_{k+1}} \left(\sum_{l=n_k+1}^{j} c_l \varphi_l \right)^4 d\mu < \infty. \tag{18}$$

Hence by the Beppo Levi theorem we have

$$\sum_{k=1}^{\infty} \max_{n_k < j \leqq n_{k+1}} \left(\sum_{l=n_k+1}^{j} c_l \varphi_l \right)^4 < \infty,$$

hence

$$\max_{n_k < j \leqq n_{k+1}} \left| \sum_{l=n_k+1}^{j} c_l \varphi_l \right| \to 0$$

almost everywhere. This fact and the almost everywhere convergence of the sequence (17) prove our theorem in the case when (15) holds.

Now Theorem 1 can be proved by induction. Suppose that, for every sequence $\{a_k\}$ and for every system $\{\chi_k\}$ having the properties (5) and (6) we have already proved that the condition

$$\sum_{k=1}^{\infty} a_k^2\, l_{r-1}^2(k) < \infty. \tag{19}$$

implies the almost everywhere convergence of the series

$$\sum_{k=1}^{\infty} a_k \chi_k(x).$$

Let $\{c_k\}$ be a sequence of real numbers for which (7) holds. Now we can construct a sequence $\{n_k\}$ for which (9) and (10) hold. Then we can obtain by the same way that σ_{n_K} (defined by (17)) is convergent almost everywhere, if we replace the reference to Theorem A by a reference to the condition (19) of our induction. (18) follows from (10).

References

[1] G. ALEXITS, *Convergence problems of orthogonal series* (Budapest, 1961).
[2] G. ALEXITS—K. TANDORI, Über das Konvergenzverhalten einer Klasse von Orthogonalreihen, *Annales Univ. Budapest*, 3 (1961), 15—18.
[3] P. RÉVÉSZ, Some remarks on strongly multiplicative systems, *Acta Math. Acad. Sci. Hung.*, **16** (1965), 441—446.

(Received June 17, 1966)

Über die Divergenz der Walshschen Reihen

Von KÁROLY TANDORI in Szeged

1. Es sei $r_n(x) = \text{sign} \sin 2^n \pi x$ die n-te Rademachersche Funktion. Das Walshsche System $\{w_n(x)\}_0^\infty$ ist folgenderweise definiert: $w_0(x) \equiv 1$ in $(0, 1)$; ist $n = 2^{\nu_1} + \ldots + 2^{\nu_p}$ $(\nu_1 < \ldots < \nu_p)$ die dyadische Darstellung von $n \geqq 1$, so sei $w_n(x) = r_{\nu_1+1}(x) r_{\nu_2+1}(x) \ldots r_{\nu_p+1}(x)$. Bekanntlich ist das Walshsche System in $(0, 1)$ orthonormiert. Aus dem Resultat von A. M. OLEVSKIJ[1]) und P. L. ULJANOV[2]) folgt, daß es eine Funktion von $L^2(0, 1)$ derart gibt, daß ihre Walshsche Entwicklung in gewisser Anordnung ihrer Glieder fast überall divergiert.

In dieser Note werden wir die folgende, schärfere Behauptung beweisen.

Satz. *Es sei* $\{\varrho(n)\}$ *eine positive Folge mit* $\varrho(n) = o(\sqrt{\log \log n})$. *Dann gibt es eine Koeffizientenfolge* $\{a_n\}$ *mit*

$$\sum a_n^2 \varrho^2(n) < \infty, \tag{1}$$

für die die Walshsche Reihe

$$\sum a_n w_n(x)$$

in gewisser Anordnung ihrer Glieder in $(0, 1)$ *fast überall divergiert.*

2. Es sei a eine positive ganze Zahl. Wir setzen

$$\varphi_a\left(\frac{l}{2^a}; x\right) = \frac{1}{2^{a-1}} \prod_{k=2}^{a} \left(1 + r_k\left(\frac{l}{2^a} + \frac{1}{2^{a+1}}\right) r_k(x)\right) \qquad (l = 0, \ldots, 2^a - 1).$$

$\varphi_a\left(\frac{l}{2^a}; x\right)$ ist die Linearkombination von Walshschen Funktionen $w_0(x), w_2(x), \ldots$ $\ldots, w_{2^a-2}(x)$; es gilt

$$\varphi_a\left(\frac{l}{2^a}; x\right) = \begin{cases} 1 & \left(\frac{l}{2^a} < x < \frac{l+1}{2^a}, \ \frac{1}{2} + \frac{l}{2^a} < x < \frac{1}{2} + \frac{l+1}{2^a} \text{ oder } \frac{l}{2^a} - \frac{1}{2} < x < \frac{l+1}{2^a} - \frac{1}{2}\right), \\ 0 & \text{sonst} \end{cases}$$

[1]) А. М. Олевский, Расходящиеся ряды из L^2 по польным системам, *Доклады Акад. Наук СССР*, **138** (1961), 545—548.

[2]) П. Л. Ульянов, Расходящиеся ряды по системе Хаара и по базисам, *Доклады Акад. Наук СССР*, **138** (1961), 556—559.

und

$$\int_0^1 \varphi_a^2\left(\frac{l}{2^a};x\right)dx=\frac{1}{2^{a-1}}.$$

Wir setzen

$$\Phi_1(0;x)=\varphi_2(0;x),$$

$$\Phi_1(1;x)=r_3(x)\varphi_2(0;x),\quad \Phi_2(1;x)=-r_3(x)r_1(x)\varphi_2(0;x),$$

$$\Phi_1(2;x)=r_4(x)\varphi_3(0;x),\quad \Phi_2(2;x)=-r_1(x)\Phi_1(2;x),$$

$$\Phi_3(x)=r_5(x)\varphi_3\left(\frac{1}{2^3};x\right),\quad \Phi_4(x)=-r_1(x)\Phi_3(x),$$

und in allgemeinen

$$\Phi_{2j+1}(k;x)=r_{2+2^{k-1}+j}(x)\sum_l \varphi_{2+2^{k-2}+[j/2]}(x_l;x)\qquad (j=0,\dots,2^{k-1}-1),^{3)}$$

wobei x_l die linksseitigen Endpunkten derjenigen Intervallen im $(0,\frac{1}{2})$ bezeichnen, in welchen $\Phi_{j+1}(k-1;x)$ positiv ist und

$$\Phi_{2j}(k;x)=-r_1(x)\Phi_{2j-1}(k;x)\qquad (j=1,\dots,2^{k-1}).$$

Es sei $m(\geqq 2)$ eine positive ganze Zahl. Für die Funktionen $\Phi_r(k;x)$ ($k=0,\dots,m-1$; $r=1,\dots,2^k$) gelten offensichtlich die folgenden Behauptungen: jede Funktion $\Phi_r(k;x)$ ist eine Linearkombination von Walshschen Funktionen; verschiedene $\Phi_r(k;x)$ haben in seinen Darstellungen keine gemeinsame Walshsche Funktion; für die Darstellungen von $\Phi_r(k;x)$ ($k=0,\dots,m-1$; $r=1,\dots,2^k$) brauchen wir nur die Walshschen Funktionen $w_0(x),\dots,w_{5.2^{2^{m-1}}-2}(x)$; es gilt

$$\sum_{r=1}^{2^k}\int_0^1 \Phi_r^2(k;x)\,dx\leqq 1.$$

Wir betrachten die Summe

$$S_m(x)=\Phi_1(0;x)+\sum_{k=1}^{m-1}\sum_{j=0}^{2^{k-1}-1}\left(\Phi_{2j+1}(k;x)+2\Phi_{2(j+1)}(k;x)\right)=$$

$$=\sum_{l=0}^{5.2^{2^{m-1}}-2} c_l(m)\,w_l(x).$$

Offensichtlich ist

$$\int_0^1 S_m^2(x)\,dx\leqq 5m.$$

---

3) $[\alpha]$ bezeichnet den ganzen Teil von α.

Wir definieren eine Anordnung der Summe $S_m(x)$. Es sei

$$S_1(x) = \Phi_1(0; x) + \Phi_1(1; x) + 2\Phi_2(1; x),$$

$$S_2(x) = \Phi_1(0; x) + \Phi_1(1; x) + \Phi_1(2; x) + 2\Phi_2(2; x) + 2\Phi_2(1; x) + \Phi_3(2; x) + 2\Phi_4(2; x),$$

u.s.w. Die Anordnung von $S_{\mu+1}(x)$ erhalten wir derart, daß wir in $S_\mu(x)$ nach dem Glied $\Phi_{2j+1}(\mu; x)$, bzw. nach dem Glied $2\Phi_{2(j+1)}(\mu; x)$ die Summe $\Phi_{2^2 j+1}(\mu+1; x) + 2\Phi_{2^2 j+2}(\mu+1; x)$, bzw. die Summe $\Phi_{2^2 j+3}(\mu+1; x) + 2\Phi_{2^2 j+4}(\mu+1; x)$ einschreiben. Nach der Definition von $\Phi_r(k; x)$ ist es klar, daß das Maximum der Partialsummen dieser Anordnung von $S_m(x)$ in den nicht-dyadischen Punkten von $(0, \frac{1}{4})$ den Wert m hat. Dieselbe Behauptung gilt für $S_m\left(x - \frac{s}{4}\right)$ $(s = 0, 1, 2, 3)$ im Intervall $(s/4, (s+1)/4)$.

Es sei endlich

$$\sigma_m\left(x - \frac{s}{4}\right) = S_m\left(x - \frac{s}{4}\right) \Big/ m.$$

Dann ist

$$\int_0^1 \sigma_m^2\left(x - \frac{s}{4}\right) dx \leqq 5/m, \tag{2}$$

und $\sigma_m\left(x - \frac{s}{4}\right)$ hat eine Anordnung ihrer Glieder, für die das Maximum der Partialsummen in den nicht-dyadischen Punkten von $(s/4, (s+1)/4)$ den Wert 1 besitzt.

3. Wir definieren eine Indexfolge $(2 \leqq) m_1 < \ldots < m_k < \ldots$ mit der folgenden Eigenschaft:

$$\varrho(n)/\sqrt{\log\log n} \leqq 1/k \qquad (n \geqq 2^{2^{m_k}}).$$

Dann ist

$$2^{2^{m_k}} + 5.2^{2^{m_k}-1} - 2 < 2^{2^{m_{k+1}}} \qquad (k = 1, 2, \ldots).$$

Wir setzen

$$\sum_{k=1}^{\infty} r_{2^{m_k}+1}(x)\, \sigma_{m_k}(x - (k-1)/4) = \sum_{k=1}^{\infty} \sum_{l=2^{2^{m_k}}}^{2^{2^{m_{k+1}}}-1} d_l(k)\, w_l(x) = \sum_{n=0}^{\infty} a_n w_n(x).$$

Nach (2) gilt offensichtlich (1). Nach dem obigen hat aber diese Reihe eine Anordnung ihrer Glieder derart, daß die angeordnete Reihe fast überall divergiert.

Damit haben wir unseren Satz bewiesen.

(Eingegangen am 7. Mai 1966)

Correction à la Note "Sur les contractions de l'espace de Hilbert. XI. Transformations unicellulaires"[1])

Par BÉLA SZ.-NAGY à Szeged et CIPRIAN FOIAŞ à Bucarest

Le raisonnement à la p. 317 de cette Note, lignes 7—11, est incomplet. On peut combler cette lacune en remplaçant les deux phrases en question ("Comme $k(\lambda)$ est ... $p(\lambda)$ n'est pas une constante") par les suivantes:

Cela exclut le cas $c = b$, $p(\lambda) = k(\lambda)$, car $N-1 \geqq 2$ et $k(\lambda)$ n'est pas une fonction constante. Ainsi on a $c=a$, $p(\lambda)=m_T(\lambda)$, donc $k(\lambda)$ est divisible par $m_T^{N-1}(\lambda)$. Vu aussi (4.20) il résulte que $k(\lambda)=m_T^{N-1}(\lambda)$, d'où, en vertu en (4.20), $\det \Omega_T(\lambda) \equiv 1$. Par suite, en formant la relation (3.3) pour $\Omega_T(\lambda)$ au lieu de $\Theta_T(\lambda)$, on obtient $\Omega_T(\lambda)\Omega_T^A(\lambda)=\Omega_T^A(\lambda)\Omega_T(\lambda)\equiv I_N$ $(\lambda \in D)$, $\Omega_T^A(\lambda)$ étant une fonction matricielle, intérieure des deux côtés. Il s'ensuit

$$\|e\| = \|\Omega_T^A(0)\Omega_T(0)e\| \leqq \|\Omega_T(0)e\| \leqq \|e\| \qquad (e \in \mathfrak{E}),$$

ce qui montre que la fonction $\Omega_T(\lambda)$ n'a pas de partie pure, donc, en vertu de la proposition 4.1 de [IX][2]), nous avons $\Omega_T(\lambda) \equiv \Omega_0$ où Ω_0 est une matrice unitaire constante. Par suite, en vertu de (3.4), $\Theta_T(\lambda)$ coïncide avec $m_T(\lambda)I_N$, ce qui entraîne que T est unitairement équivalente à S^N. Or, comme $N>1$, S^N n'est pas unicellulaire tandis que T l'est par l'hypothèse faite. Cette contradiction montre que le cas $c=a$ est aussi impossible.

(Reçu le 10. novembre 1966)

[1]) *Acta Sci. Math.*, **26** (1965), 301—324.
[2]) *Acta Sci. Math.*, **25** (1964), 283—316.

Bibliographie

G. Pólya, Mathematik und plausibles Schließen, Bd. 1. Induktion und Analogie in der Mathematik, 403 Seiten; Bd. 2. **Typen und Strukturen plausibler Folgerung,** 282 Seiten (Sammlung „Wissenschaft und Kultur", Bd. 14 und 15), Basel—Stuttgart, Birkhäuser Verlag, 1962—63. Ins deutsche übersetzt von LULU BECHTOLSHEIM.

Die in letzter Zeit erschienenen Bücher von G. PÓLYA fanden nicht nur bei den in der Forschung tätigen Mathematikern, sondern auch bei den Mathematiklehrern aller Stufen begeisterte Aufnahme. Der Autor gibt außerordentlich wertvolle Hinweise zur Gestaltung anspruchsvollen mathematischen Unterrichtes, der nicht nur zeitgemäße Kenntnisse vermittelt, sondern auch zu weiterführenden Überlegungen anregt. Eine solche pädagogische Zielstellung besaß — und erfüllte in hervorragender Weise — z. B. seine gemeinsam mit G. SZEGŐ verfaßte bekannte Aufgabensammlung *(Aufgaben und Lehrsätze aus der Analysis)*. Für die Einschätzung des vorliegenden Buches ist besonders bemerkenswert, daß der Verfasser zu den aufgeworfenen Fragen auch aus philosophischer Sicht Stellung nimmt.

Z. B. äußert er folgende Gedanken: Streng genommen besteht unser ganzes Wissen außerhalb der Mathematik und demonstrativen Logik (die ja in der Tat ein Zweig der Mathematik ist) aus Vermutungen. Wir sichern die Gültigkeit unseres mathematischen Wissens durch demonstratives Schließen, aber wir unterstützen unsere Vermutungen durch plausibles Schließen. Plausibles Schließen ist gewagt, strittig und provisorisch. Demonstratives Schließen ist sicher, unbestreitbar und endgültig. Die Mathematik wird als eine demonstrative Wissenschaft angesehen. Aber die im Entstehen begriffene Mathematik gleicht jeder anderen Art menschlichen Wissens, das im Entstehen ist. Das Resultat der schöpferischen Tätigkeit des Mathematikers ist demonstratives Schließen: der Beweis. Der Beweis wird aber durch plausibles Schließen entdeckt: durch Erraten. Wenn das Erlernen der Mathematik einigermaßen ihre Erfindung widerspiegeln soll, so muß es einen Platz für das Erraten, für das plausible Schließen haben.

Aus der Vielzahl der im ersten Band behandelten schönen Themen sei insbesondere auf die folgenden verwiesen: die Goldbachsche Vermutung, der Eulersche Polyedersatz, das Bachet-Problem, einige Reihensummationen von Euler, Summenfunktion aller Teiler einer ganzen Zahl. Man betrachtet auch viele kleinere, aber interessante Besonderheiten.

Die sehr originelle Aufgabensammlung verdient besonders hervorgehoben zu werden: insbesondere erhellt die Reihenfolge der Aufgaben gewisse eigentümliche Forschungsmethoden.

Im zweiten Band erweitert der Verfasser seine philosophischen Ausführungen und weist auf Anwendungen zahlreicher struktureller, logischer und halblogischer Methoden hin. Hierbei bedient sich der Autor besonders der Wahrscheinlichkeitsrechnung. Außerdem werden in diesem Band analoge Probleme anderer Wissenschaftszweige dargestellt. Äußerst lehrreich sind z. B. die folgenden Bemerkungen des Verfassers in dem Abschnitt „Ein paar Worte an den Lehrer" (S. 240).: „Laßt uns erraten lehren. Ich sagte, daß es wünschenswert sei, erraten zu lehren, aber nicht, daß es leicht sei. Dieses Buch wendet sich in erster Linie an Studierende der Mathematik, die ihre eigenen Fähigkeiten entwickeln wollen und an Leser, denen daran gelegen ist, etwas über plausibles Schließen und seine nicht ganz banale Beziehung zur Mathematik zu lernen."

Der Erfolg des Buches ist ein Beweis dafür, daß der prominente Verfasser dem oben genannten Anliegen in hervorragender Weise gerecht geworden ist.

Die Ausstattung des Buches ist vorzüglich.

J. Berkes (Szeged)

G. Pólya, Les mathématiques et le raisonnement „plausible", 299 pages, Paris, Gauthier—Villars, 1958. Ins französische übersetzt von ROBERT VALLÉE.

Das Buch ist die erste fremdsprachige Übersetzung des in Princeton und London erschienenen Werkes von G. PÓLYA: *Mathematics and plausible reasoning.* Diese Ausgabe unterscheidet sich von der oben besprochenen deutschen Übersetzung nur darin, daß auf die Aufgabensammlung verzichtet und die Ausgabe in einen Band vorgenommen wurde.

J. Berkes (Szeged)

A. Kaufmann—R. Douriaux, Les fonctions de la variable complexe, VIII+428 pages, 355 figures et 7 tableaux, Paris, Editions Eyrolles et Gauthiers—Villars, 1962.

Comme le sous-titre du livre l'indique ("Théorie et applications au niveau de l'ingénieur"), les auteurs s'adressent surtout aux ingénieurs, en présentant un grand nombre d'exemples et d'exercices pris dans la physique et technique (concernant l'électricité théorique, la mécanique des fluides, la chaleur, les réseaux électriques, etc.). Dans les démonstrations il y a parfois quelques lacunes (comme par exemple dans celle du théorème fondamental de l'algèbre), mais les applications indiquées peuvent être d'intérêt aussi pour les mathématiciens.

Béla Sz.-Nagy (Szeged)

R. Courant und H. Robbins, Was ist Mathematik? XVI+399 Seiten, Springer-Verlag, Berlin—Göttingen—Heidelberg, 1962.

Although mathematics seems to have a more important role in our life then ever before, there are very few people who have a real understanding of mathematics and can imagine what a great role is played by mathematics in various branches of modern sciences. It is well-known that there is a danger that sometimes students get only a formal routine of manipulations without the real understanding of the mathematical concepts. Therefore it is warmly to greet the appearance of such books by the aid of which the danger mentioned above can be avoided.

This book is a translation of the highly successful Englisch original, first publisched in 1941. It is written for a wide class of readers: students, teachers, engineers and so its style is popular in some sense, but as Professor Courant says in the preface of the first edition: "It requires a certain degree of intellectual maturity and a willingness to do some thinking on one's own." Perhaps this sentence is the most characteristic for this book. The chapters which follow in a systematic order, are in some sense independent of one another. In each chapter, after a clear introduction, the path goes gradually to the root of the matter with the most active collaboration of the reader. The great pedagogical experience of the authors assures that the reader setting out from an elementary level can obtain an insight into the very essence of mathematics.

The contents are grouped as follows: The Natural Numbers; The Number System of Mathematics; Geometrical Constructions; The Algebra of Number Fields; Projective Geometry; Axiomatics; Non-Euclidean Geometries; Topology; Functions and Limits; Maxima und Minima; The Calculus. There is an Appendix containing supplementary remarks, problems, and exercises.

L. Pintér (Szeged)

Lucy Joan Slater, Generalized Hypergeometric Functions, XIII+273 pages, Cambridge, University Press, 1966.

Seit dem Erscheinen der Monographie von W. N. BAILEY: *Generalized Hypergeometric Series* (Cambridge Mathematical Tract, No. 32. 1935; zweite Auflage 1964) blieb diese Theorie immer im Vordergrund der Forschungen über spezielle Funktionen. Die Verfasserin, die eine Mitarbeiterin von BAILLEY war und ihre Arbeit dem Gedächtnis ihres unlängst verstorbenen Lehrers widmet, gibt im vorliegenden Buch eine wesentlich erweiterte, zusammenfassende Darstellung, wobei die diesbezüglichen Resultate der letzten Jahrzehnte und darunter natürlich auch die eigenen Forschungsergebnisse der Verf. vielseitig einbezogen werden.

Während im ersten Kapitel die klassische Theorie der Gaußschen hypergeometrischen Reihe

$$1+\frac{ab}{c}\frac{z}{1!}+\frac{a(a+1)\cdot b(b+1)}{c(c+1)}\frac{z^2}{2!}+\dots$$

behandelt wird, untersucht man im folgenden die Eigenschaften der verallgemeinerten Gaußschen Funktion

$$ {}_AF_B[(a);(b);z]=\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\dots(a_A)_n}{(b_1)_n(b_2)_n\dots(b_B)_n}\cdot\frac{z^n}{n!} $$

mit $(a)_n = a(a+1)\dots(a+n-1)$ und ihre sogenannten q-Analoga („basic hypergeometric functions"), welche mittels Produkten der Form $(a;q)_n = (1-a)(1-aq)\dots(1-aq^{n-1})$ $(|q|<1)$ anstatt von $(a)_n$ gebildet sind (Kap. 2—3). ${}_AF_B[(a);(b);z]$ enthält bekanntlich alle gewöhnlich benutzten Funktionen der Analysis als Spezialfälle und deren Theorie ist — neben ihrer funktionentheoretischen und statistischen Bedeutung — ein wertvolles Hilfsmittel zur Lösung komplizierterer Differentialgleichungen der mathematischen Physik. Was die q-Erweiterungen anbetrifft, so haben diese tiefliegende Zusammenhänge mit der Theorie der elliptischen Funktionen und viele Anwendungsmöglichkeiten in der Zahlentheorie, namentlich bezüglich gewisser Partitionsprobleme. Die Diskussion der verallgemeinerten hypergeometrischen Integrale wird im vierten und fünften Kapitel sehr sorgfältig durchgeführt, und zwar unter Heranziehung neuerer Integralsätze und asymptotischer Resultate von MacRobert und Meijer, ferner von Sears und Slater. Die Kapitel 6—9 sind weiteren Verallgemeinerungen, nämlich bilateralen hypergeometrischen Reihen und dem Fall von mehreren Veränderlichen (Appellsche bzw. Lauricellasche Reihen) gewidmet. Aus dem reichen Inhalt dieses Teils seien die Anwendungen über Jacobische Thetafunktionen und Identitäten vom Rogers—Ramanujanschen Typ hervorgehoben. — Der Text ist mit vielen historischen bzw. persönlichen Bemerkungen der Verfasserin durchwoben. Sechs Anhänge (Formelsammlungen, numerische Tafeln) und ein die meisten der zwischen 1934 und 1961 publizierten diesbezüglichen Arbeiten enthaltendes Literaturverzeichnis sind beigefügt.

Dieses klar abgefaßte Buch ist nicht nur für Fachleute des Gebietes von großem Nutzen, sondern es wird gewiß allen Mathematikern und Physikern behilflich sein, die sich in der sehr weitverzweigten neueren Literatur über hypergeometrische Funktionen und Reihen orientieren wollen. Bemerkt sei noch, daß einige interessante Ergebnisse der letzten Jahre und eine kleinere Monographie von R. P. Agarwall (*Generalized Hypergeometric Series*, Asia Publ. House, New York—Bombay, 1963) bei der Arbeit am Manuskript offenbar nicht berücksichtigt werden konnten; eine hoffentlich folgende zweite Auflage wird aber Erweiterungsmöglichkeiten bieten.

M. Mikolás (Budapest)

LIVRES REÇUS PAR LA RÉDACTION

N. I. Achieser—I. M. Glasmann, Theorie der linearen Operatoren im Hilbert-Raum (Mathematische Lehrbücher und Monographien, I. Abteilung, Mathematische Lehrbücher, Bd. 4), 4. unveränderte Aufl., XIII+369 Seiten, Berlin, Akademie-Verlag, 1965. — MDN 28,—

N. Bourbaki, Éléments de mathématique. Fasc. XV. Espaces vectoriels topologiques. Chapitre 1—2. **Espaces vectoriels topologiques sur un corps valué. Ensembles convexes et espaces localement convexes,** 2. éd., revue, corrigée, augmentée, VI+178 pages, Paris, Hermann, 1966. — 45 fr.

——— **Éléments de mathématique. Fasc. XXII. Théorie des ensembles.** Chapitre 4. **Structures,** 105 pages, Paris, Hermann, 1966. — 36 fr.

J. J. Burckhardt, Die Bewegungsgruppen der Kristallographie (Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften, Mineralogisch—Geotechnische Reihe, Bd. 2), 2. neuarb. Aufl., 209 Seiten, Basel—Stuttgart, Birkhäuser, 1966.

E. Čech, Topological spaces, 893 pages, Prague, Publishing House of the Czechoslovak Academy of Sciences, 1966.

L. Fuchs, Teilweise geordnete algebraische Strukturen, 311 Seiten, Budapest, Akadémiai Kiadó, 1966.

N. Gastinel, Analyse numérique linéaire (Collection Enseignement des sciences), XII+364 pages, Paris, Hermann, 1966. — 60 fr.

H. Halberstam—K. F. Roth, Sequences, Vol. 1, XX+291 pages, Oxford, Clarendon Press, 1966. — 63 s.

B. Kerékjártó, Les fondements de la géométrie, Tome 2. **Géométrie projective,** 528 pages, Budapest, Akadémiai Kiadó, 1966.

E. Leimanis, The general problem of the motion of coupled rigid bodies about a fixed point (Springer Tracts in Natural Philosophy, Vol. 7), XVI+337 pages, Berlin—Heidelberg—New York, 1965. — DM 48,—

Ch. B. Morrey, Jr., Multiple Integrals in the calculus of variations (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Bd. 130), XII+506 pages, Berlin—Heidelberg—New York, 1966. — DM 78,—

G. Pólya, Vom Lösen mathematischer Aufgaben, Einsicht und Entdeckung, Lernen und Lehren, Bd. I (Sammlung Wissenschaft und Kultur, Bd. 20), 315 Seiten, Basel—Stuttgart, Birkhäuser, 1966. — s Fr. 38,—

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Felelős szerkesztő és kiadó: Szőkefalvi-Nagy Béla
A kézirat nyomdába érkezett: 1966. július hó
Megjelenés: 1966. december hó

Példányszám: 825. Terjedelem: 12,6 (A/5) ív
Készült monószedéssel, íves magasnyomással, az MSZ 5601-54 és az MSZ 5602-55 szabvány szerint